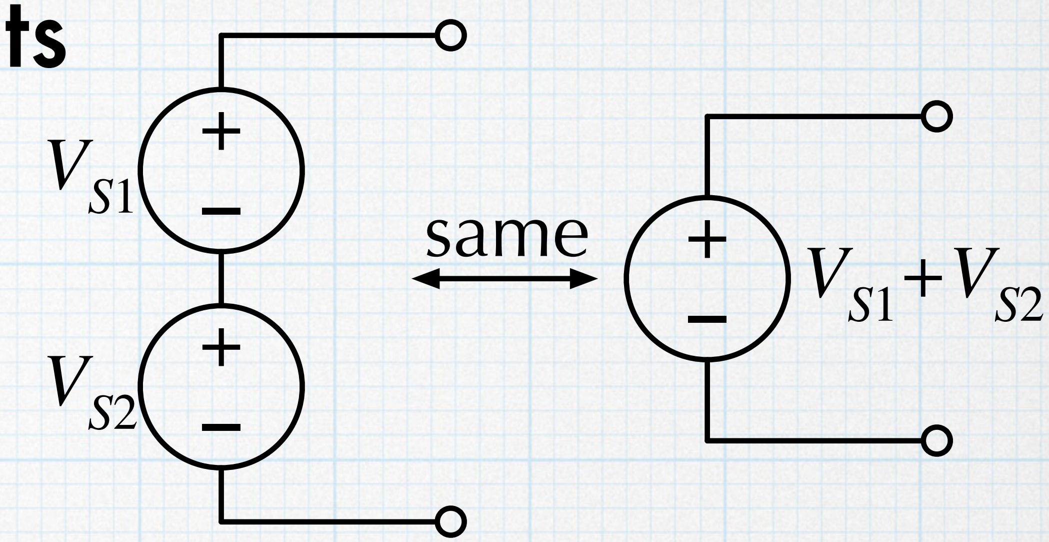
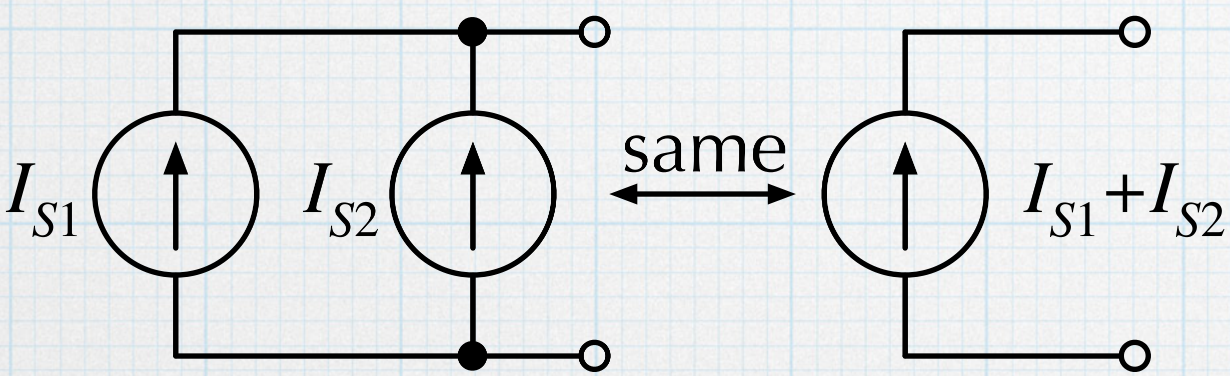
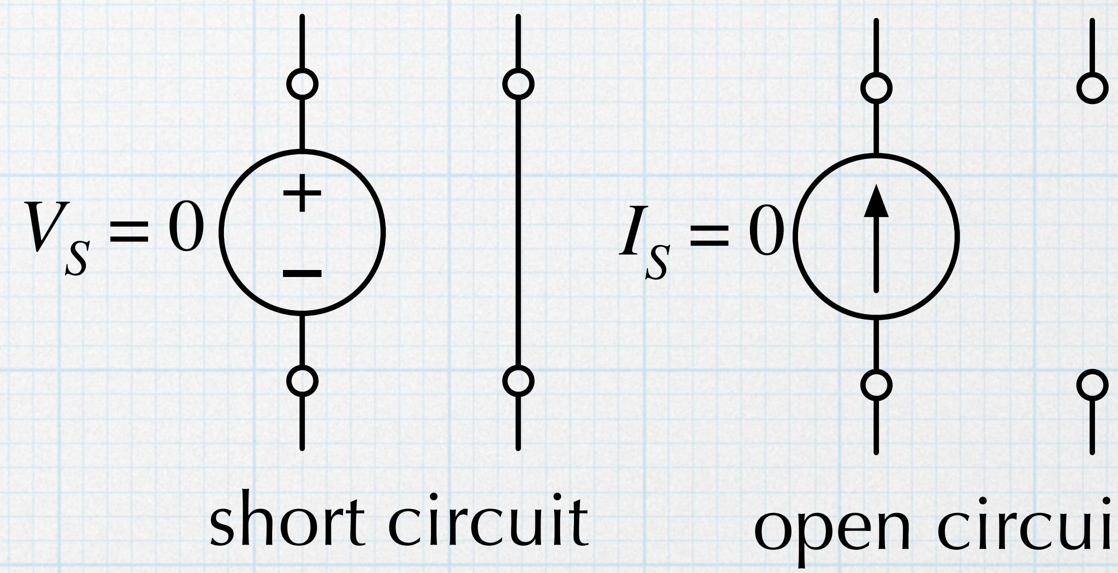
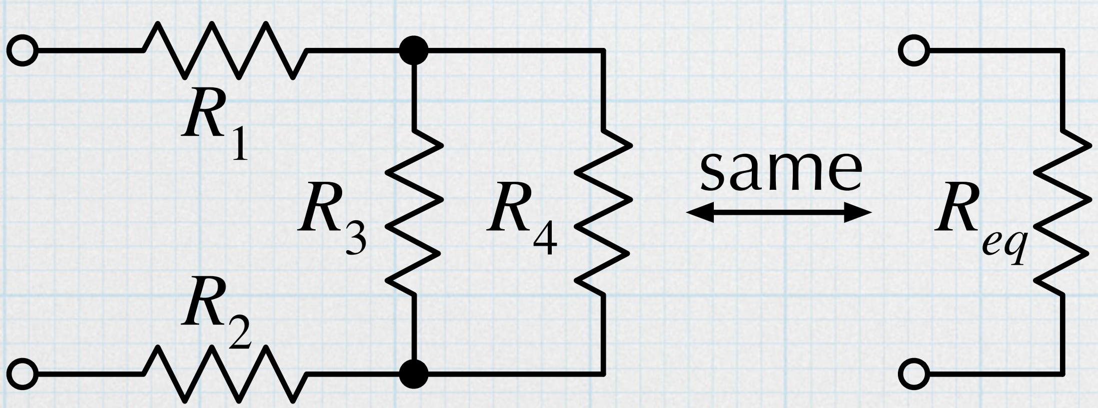
Thevenin / Norton equivalent circuits

We have seen many instances where we can take elements in a part of a circuit and combine them in some fashion to make an *equivalent* circuit. With respect to the two terminals, the two versions behave identically. Anything attaching to the two terminals will not be able to tell the difference.

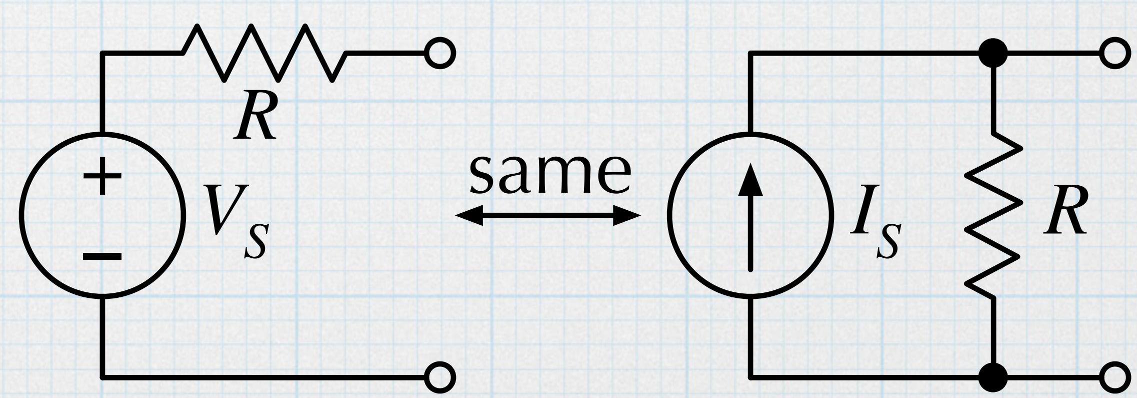








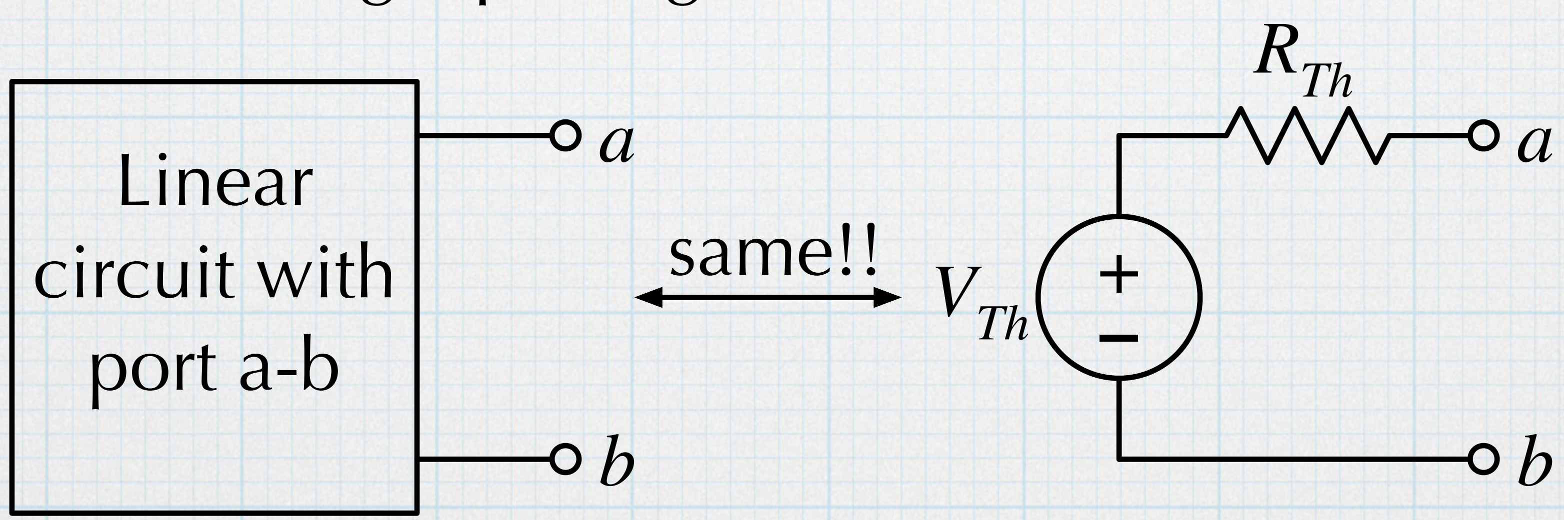
$$R_{eq} = R_1 + R_2 + R_3 \| R_4$$



$$V_S = I_S R$$

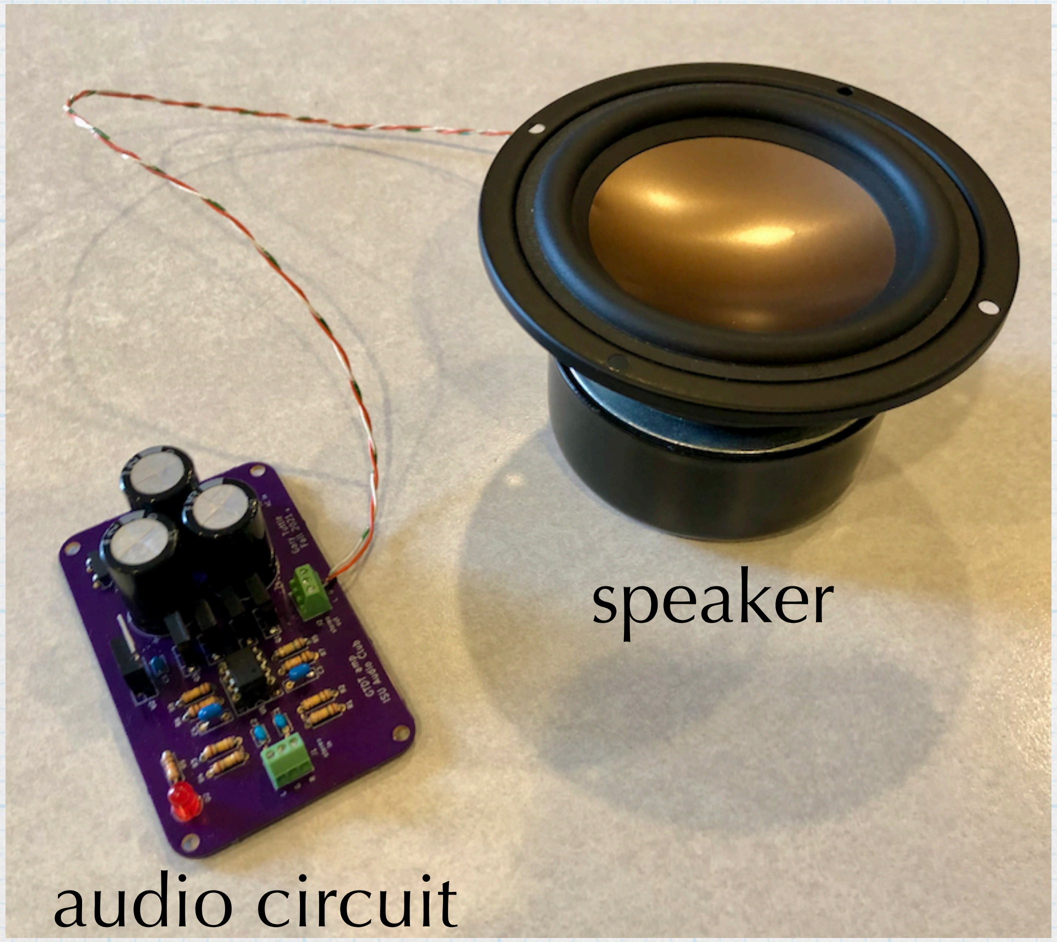
We can generalize this idea of equivalency by saying that *any* linear circuit that has a connection defined by two nodes — a port — can be simplified to an equivalent circuit consisting of a voltage source and a resistor in series. This remarkable result was proven by French engineer Leon Thevenin in 1883, and so we call the simplified voltage-source / resistor combination the *Thevenin equivalent*.

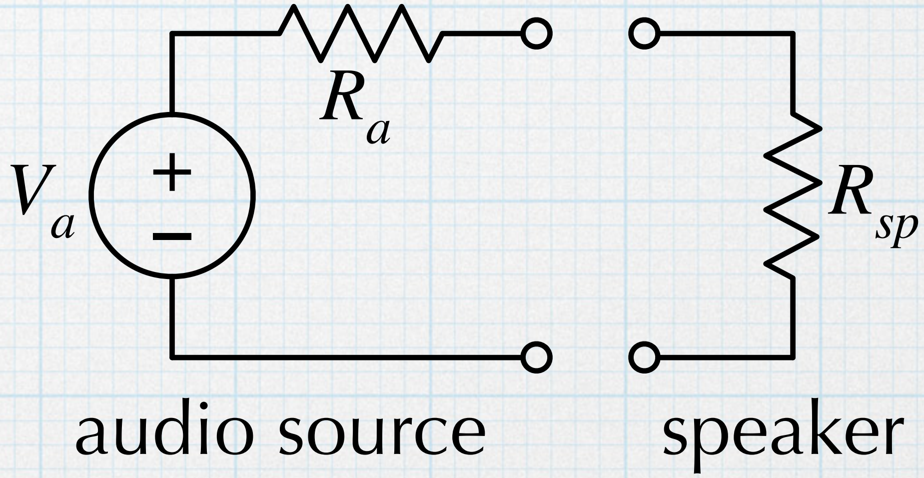
Terminology: A port is defined by two nodes in a circuit. In principle, the port could be any random pair of nodes, but usually the port represents a place where we intend to connect another circuit. Common examples are the ubiquitous 3.5-mm stereo audio jack and the even more ubiquitous USB plug. (Actually, both examples have *multiple* ports in a single package, but never mind that for now.)

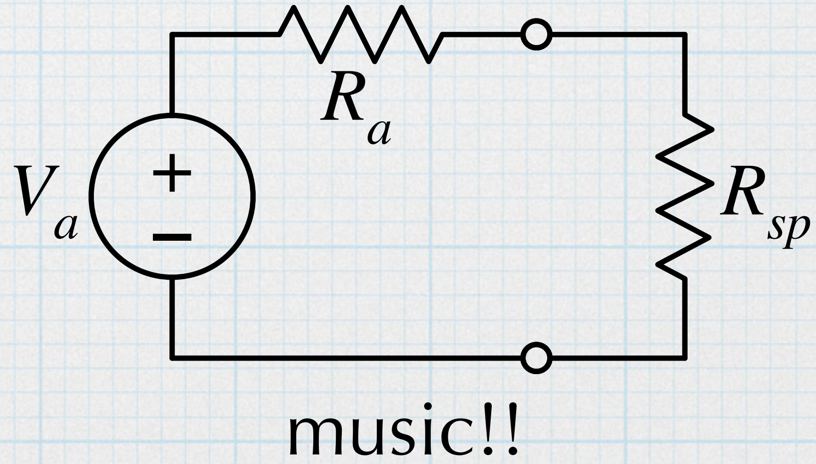


This view that every linear circuit has an effective voltage (which could be zero!) and an effective resistance completely changes the way that we look at circuits.

If we consider the output port of some audio generating gizmo — a phone or whatever — and want to hear the sound, we must connect a speaker of some sort — headphones or a loudspeaker on the shelf. If we look at the circuitry that generates the audio signal, it appears rather bewildering to a novice. Yet, Thevenin says that all of that complication can be boiled down to two components. And the speaker itself has a Thevenin equivalent, which is just a single resistor, since the speaker does not generate a voltage on its own.

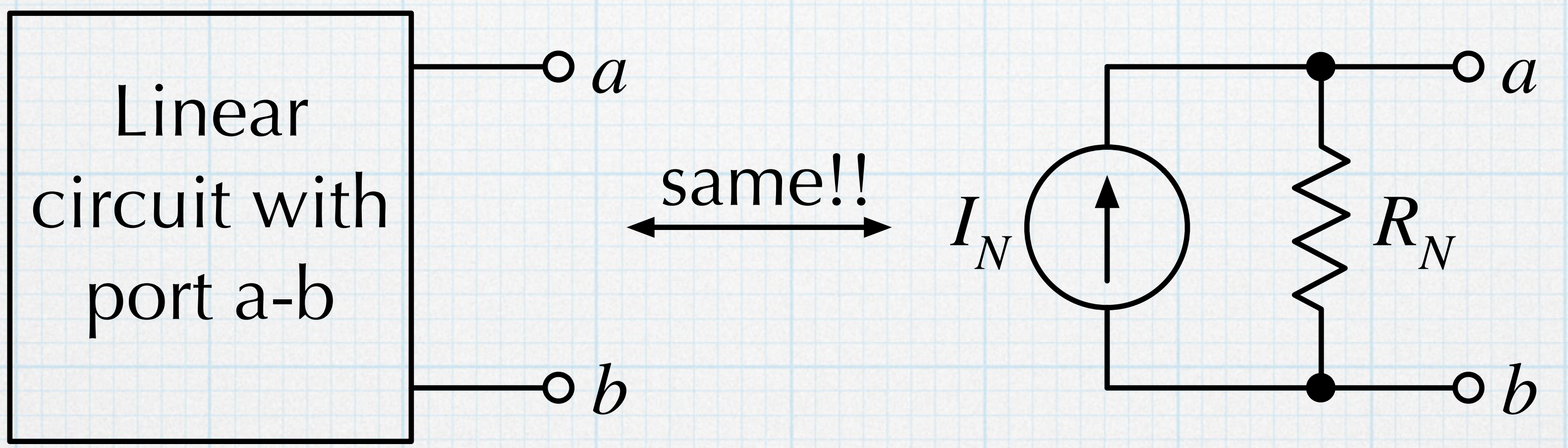






Norton equivalent

Sometime later (c. 1926), a similar equivalency idea was put forth by Edward Norton working at Bell Labs. His idea was nearly identical to Thevenin's, but Norton used a parallel combination of current source and resistor, rather than Thevenin's series arrangement. (Apparently Norton was not aware of Thevenin's earlier proof.)



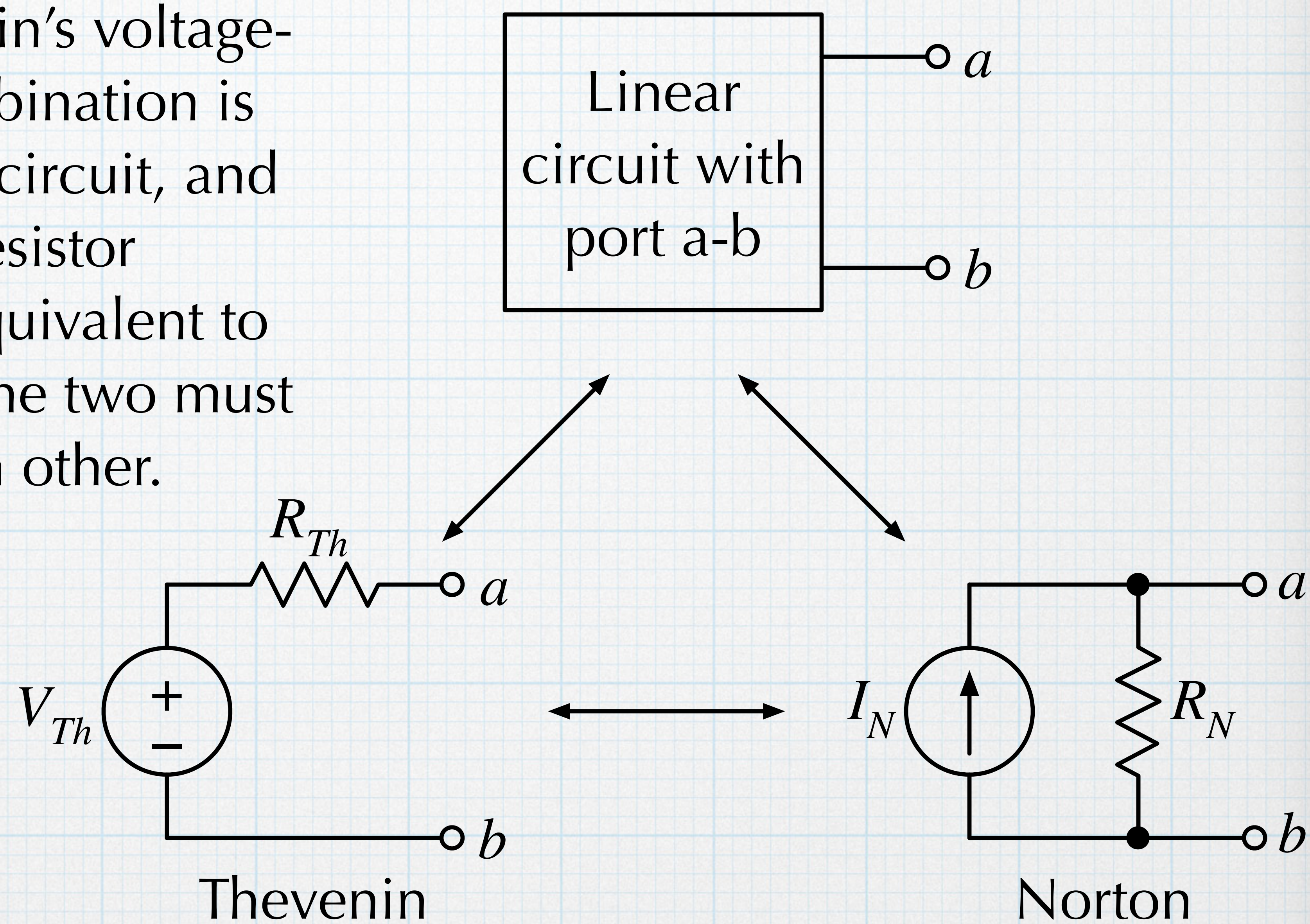
Having seen source transformations earlier, this is not surprising to us — Norton is simply the source transformation of Thevenin. And vice-versa.

We might recall that we never actually *proved* that source transformations were valid. We inferred that they were equivalent using examples.

Using Thevenin and Norton, we have proof of the validity of source transformations. If Thevenin's voltage-source/resistor series combination is equivalent to the original circuit, and Norton's current-source/resistor parallel combination is equivalent to the original circuit, then the two must also be equivalent to each other.

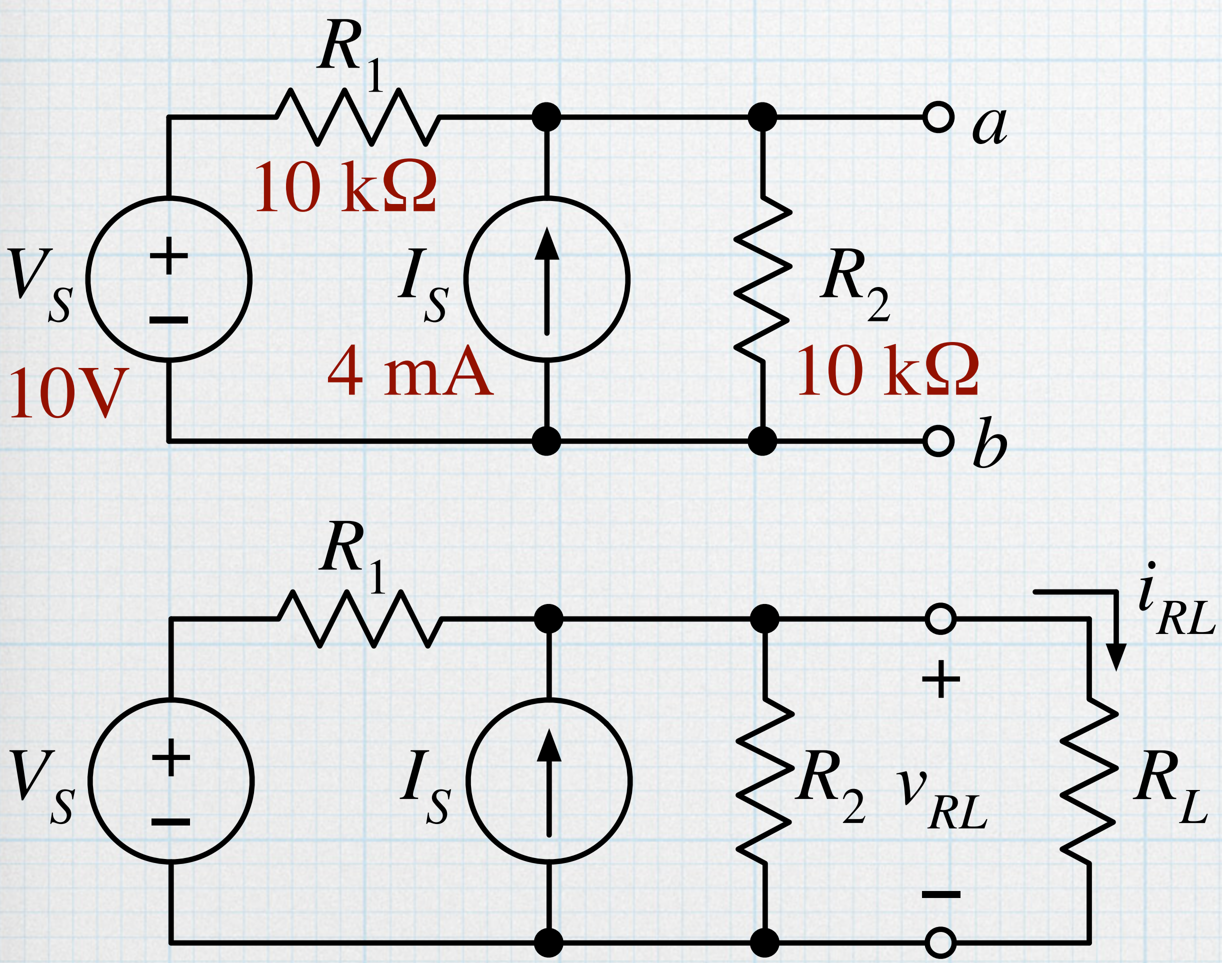
$$V_{Th} = I_n R_n$$

$$R_{Th} = R_n$$



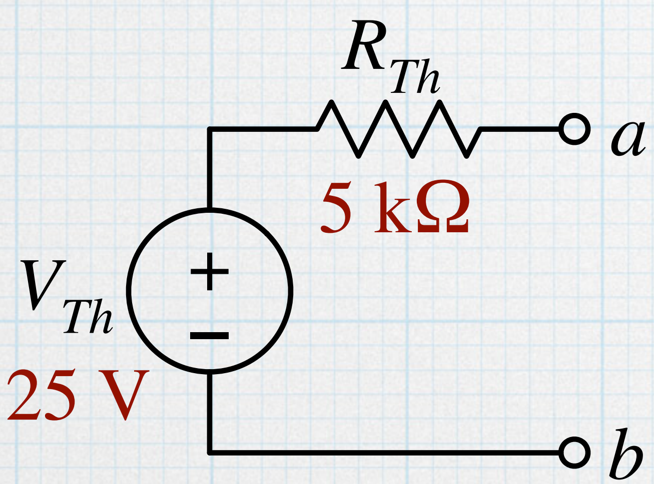
Some texts make a big deal about differentiating between Thevenin and Norton. We will take the view that they are simply two manifestations of the same idea. As soon as we know one equivalent circuit, we immediately know the other through source transformation, and in any particular case, we use the version that is most suitable to the problem.

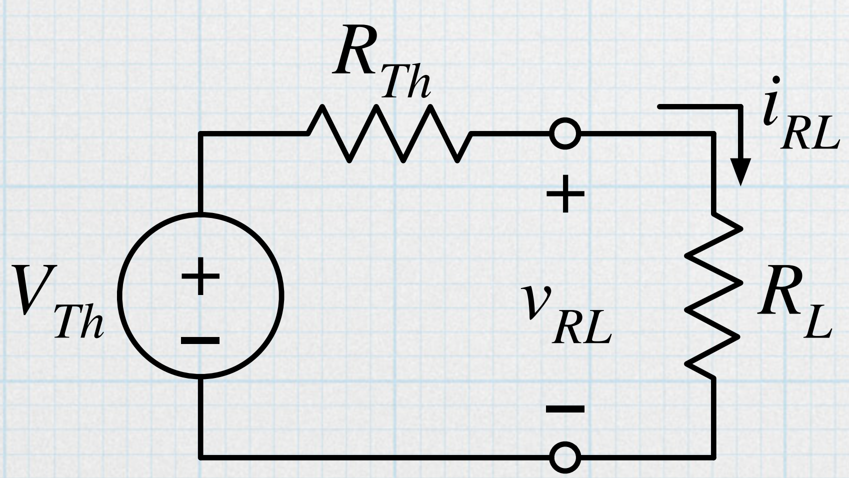
To further illustrate the equivalence idea, consider the circuit shown below — basically, it is the familiar "two-source, two-resistor" circuit with a port defined by the two nodes *a* and *b*. Attach various *load* resistors to the port, and calculate the resulting voltage across the load along with the current and power. The results are given in the table.



R_L	v_{RL}	i_{RL}	P_{RL}
50 Ω	0.2475 V	4.95 mA	1.225 mW
500 Ω	2.273 V	4.55 mA	10.33 mW
5 kΩ	12.5 V	2.5 mA	31.25 mW
50 kΩ	22.73 V	0.455 mA	10.33 mW
500 kΩ	24.75 V	49.5 μA	1.225 mW

Now consider the simple source and resistor circuit shown below. Attach the same load resistors and calculate the resulting voltages, currents, and powers. The results are shown in the table. The results from this simple circuit are identical to those from the circuit on the previous page. In terms of a load that is attached at the port, the two circuits are indistinguishable. We can use the equivalent in place of the original and have identical results. This is the idea that was proven by Thevenin.





R_L	v_{RL}	i_{RL}	P_{RL}
50 Ω	0.2475 V	4.95 mA	1.225 mW
500 Ω	2.273 V	4.55 mA	10.33 mW
5 kΩ	12.5 V	2.5 mA	31.25 mW
50 kΩ	22.73 V	0.455 mA	10.33 mW
500 kΩ	24.75 V	49.5 μA	1.225 mW

Determining the Thevenin/Norton values

Now comes a crucial question: Given a "black box circuit", how do we determine V_{Th} and R_{Th}? (Or I_N and R_N?)

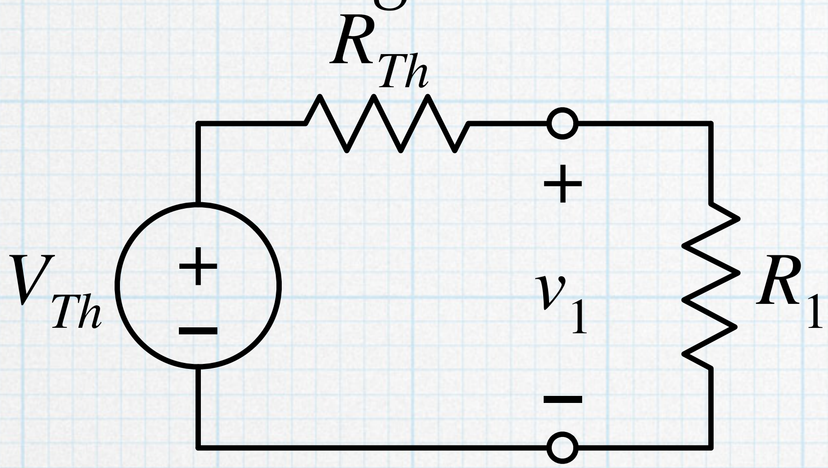
Since there are two components in an equivalent circuit, we will need to do two measurements (if working in the lab) or two calculations (if working with pencil and paper) and then determine the Thevenin or Norton values from those results. There is no easy way out — there must be two independent measurements/calculations. Fortunately, we have already learned the analysis and lab skills needed.

We use the Thevenin equivalent as a guide in determining the process. The technique involves nothing more than attaching two different loads to the port and then measuring or calculating the resulting voltages or currents at the port.

We will illustrate the idea using random values for the two loads. Then we will sharpen the technique by having the port be open circuited and then short circuited.

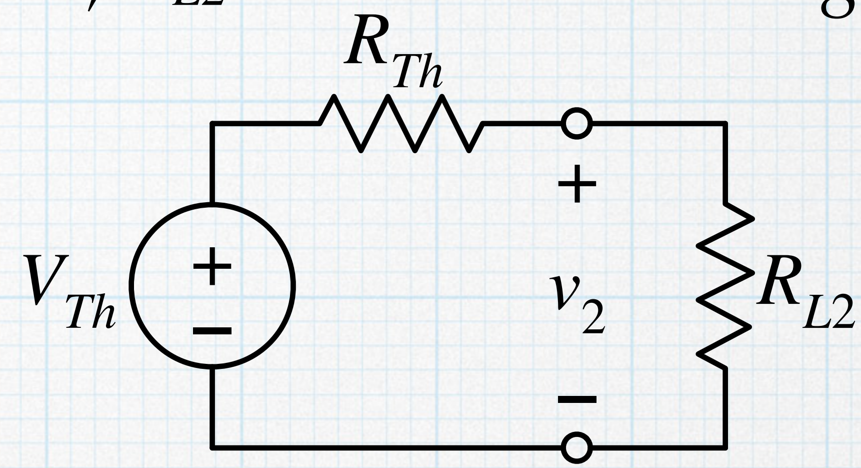
First consider what happens with a simple source + resistor circuit in which we don't know the values of the source or the resistor.

1. Attach a load resistor, R_{L1}. Measure resulting v_1.



$$v_1 = \frac{R_{L1}}{R_{L1} + R_{Th}} V_{Th}$$

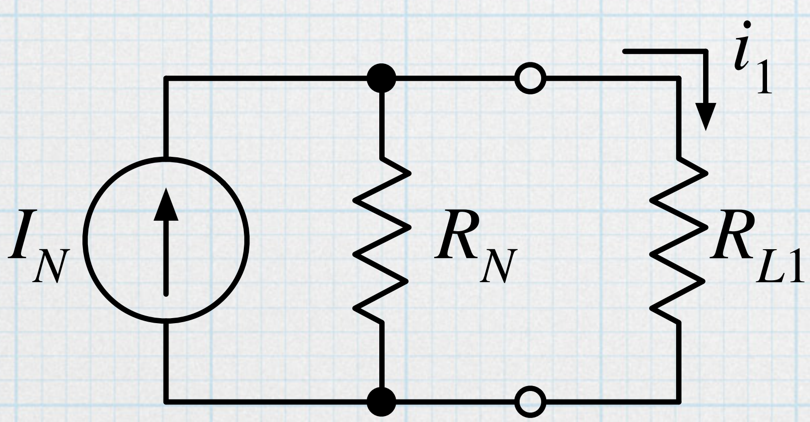
2. Attach another, different load resistor, R_{L2}. Measure resulting v_2.



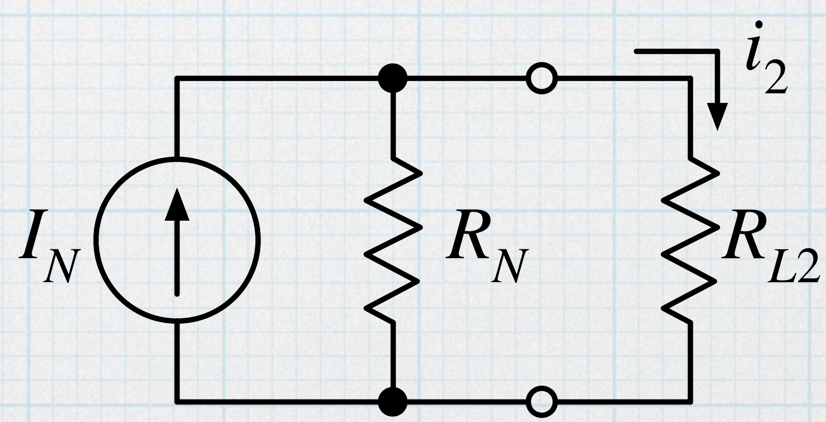
$$v_2 = \frac{R_{L2}}{R_{L2} + R_{Th}} V_{Th}$$

Knowing v_1 and v_2, it is a simple matter to use the two equations to solve for the two unknowns, V_{Th} and R_{Th}.

We could also use the same approach with a Norton equivalent:

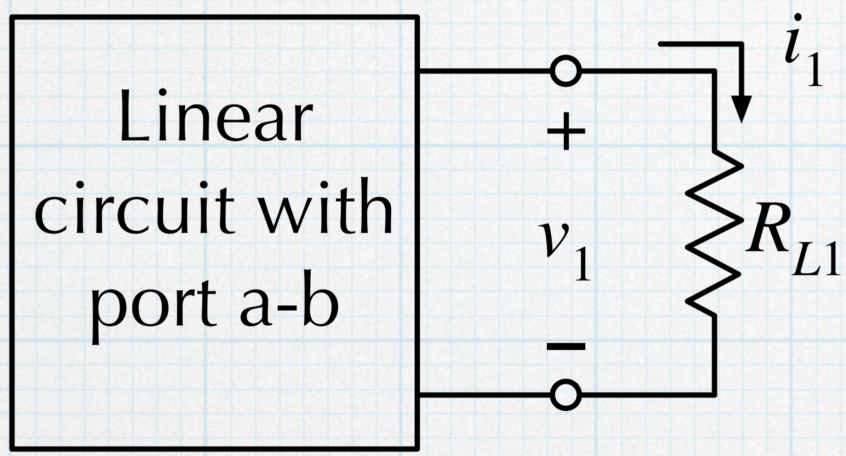


$$i_1 = \frac{\frac{1}{R_{L1}}}{\frac{1}{R_{L1}} + \frac{1}{R_N}} I_N$$



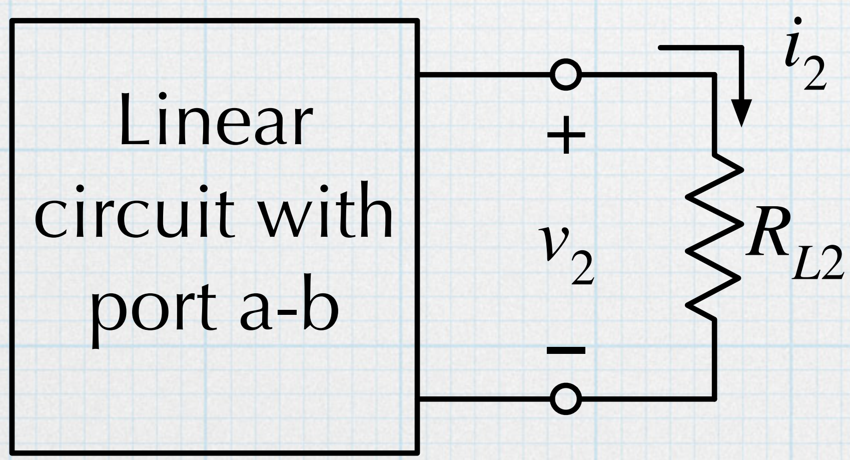
$$i_2 = \frac{\frac{1}{R_{L2}}}{\frac{1}{R_{L2}} + \frac{1}{R_N}} I_N$$

Now that we see the basic approach, we can apply it to other circuits. Attach a known load resistance and determine the corresponding port voltage. Then attach a second load and find the port voltage for that. Use the results to find the Thevenin (or Norton) equivalent for the circuit.



$$v_1 = \frac{R_{L1}}{R_{L1} + R_{Th}} V_{Th}$$

$$V_{Th} = \frac{v_1 v_2 \left(R_{L1} - R_{L2}\right)}{R_{L1} v_2 - R_{L2} v_1} = \frac{i_2 v_1 - i_1 v_2}{i_2 - i_1}$$



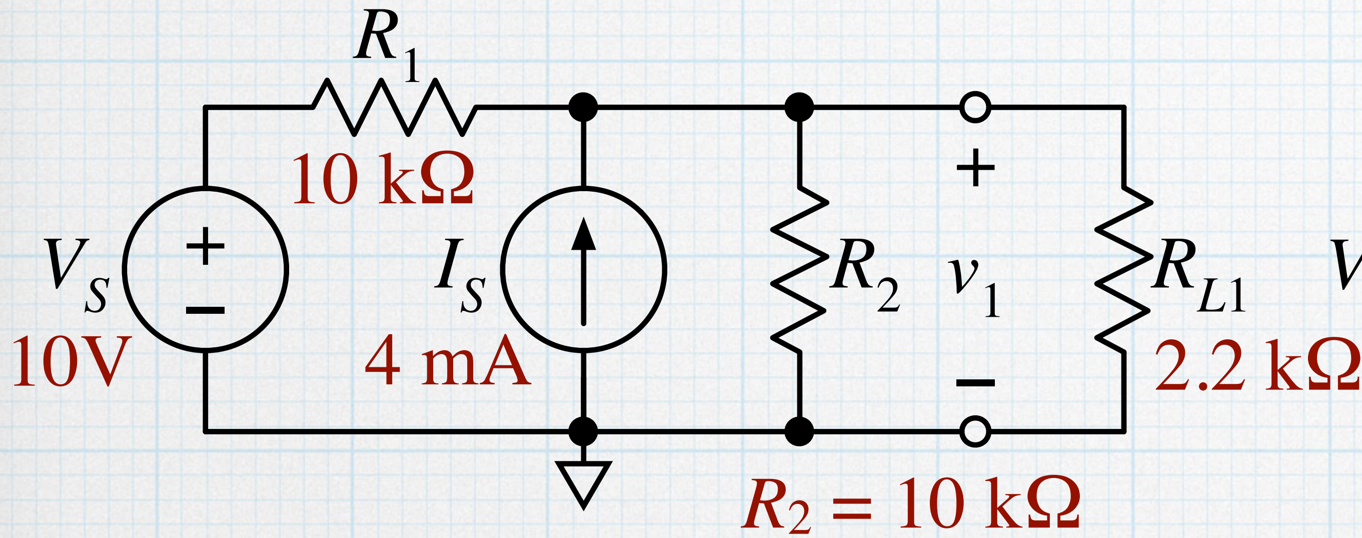
$$v_2 = \frac{R_{L2}}{R_{L2} + R_{Th}} V_{Th}$$

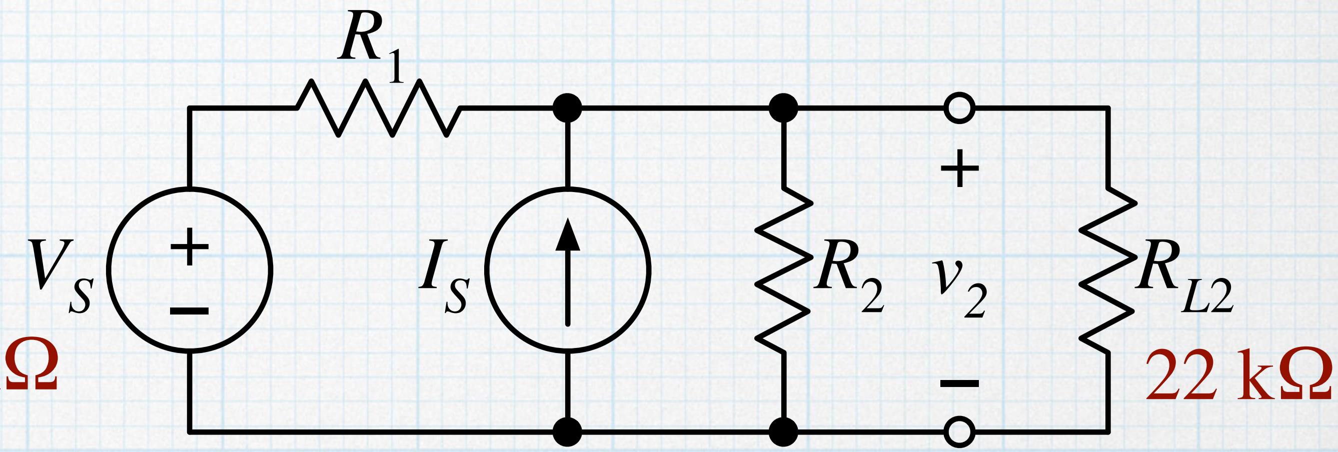
$$R_{Th} = \frac{R_{L1} R_{L2} \left(v_1 - v_2\right)}{R_{L1} v_2 - R_{L2} v_1} = \frac{v_1 - v_2}{i_2 - i_1}$$

Again, this also works using the Norton model.

Example 1

Use two different load resistors (say 2.2kΩ and 22 kΩ) to determine the Thevenin equivalent of the two-source, two-resistor circuit shown earlier on page 6. Confirm that the Thevenin equivalent on page 7 is correct.





Try node voltage:

$$\frac{V_S - v_1}{R_1} + I_S = \frac{v_1}{R_2} + \frac{v_1}{R_{L1}}$$

$$v_1 = \frac{V_s + I_S R_1}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_{L1}}} = 7.639\text{ V}$$

Eh, try superposition this time:

$$v_2' = \frac{R_2 \| R_{L2}}{R_1 + R_2 \| R_{L2}} V_S = 4.074\text{ V}$$

$$v_2'' = I_S \left(R_1 \| R_2 \| R_{L2}\right) = 16.30\text{ V}$$

$$v_2 = 20.37\text{ V}$$

Then, $V_{Th} = \dfrac{v_1 v_2 \left(R_{L1} - R_{L2}\right)}{R_{L1} v_2 - R_{L2} v_1} = 25\text{ V}$ and $R_{Th} = \dfrac{R_{L1} R_{L2} \left(v_1 - v_2\right)}{R_{L1} v_2 - R_{L2} v_1} = 5\text{ k}\Omega$

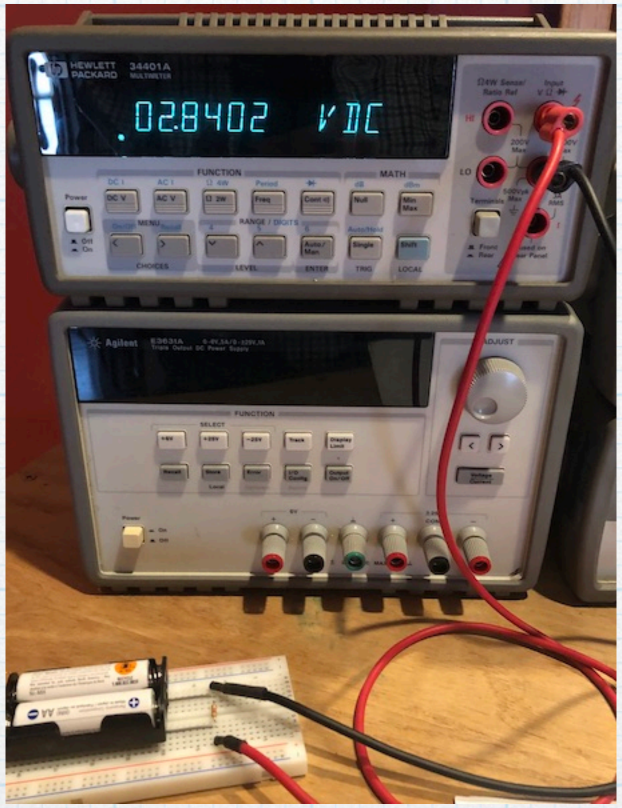
Confirmed!

Example 2

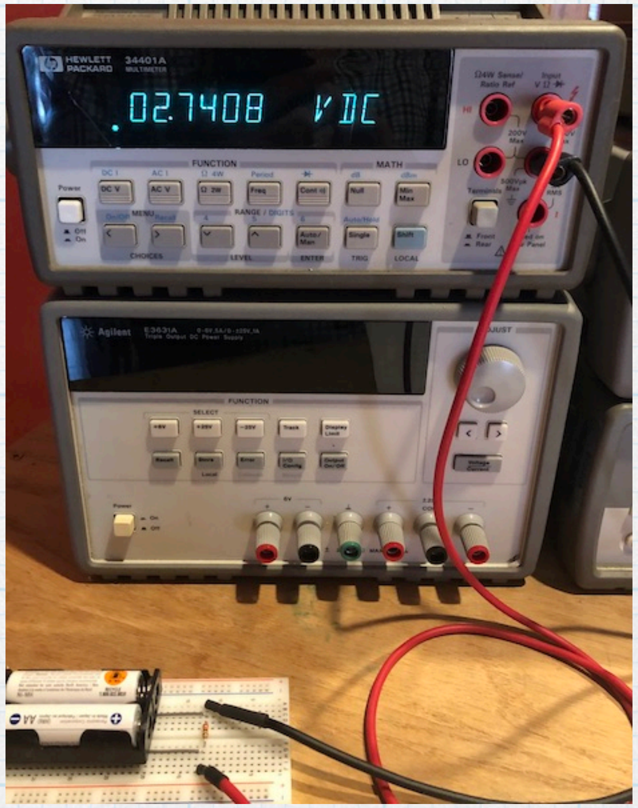
One lesson that we learn from Thevenin is that *everything* has some equivalent resistance with it — even things that we thought of as being "perfect" voltage sources. Let's measure the Thevenin equivalent of a pair of rechargeable NiMH batteries connected in series[1].

Two batteries in series.

$R_{L1} = 100\ \Omega$.



$R_{L2} = 22\ \Omega$.



$$V_{Th} = \frac{v_1 v_2 \left(R_{L1} - R_{L2}\right)}{R_{L1} v_2 - R_{L2} v_1} = 2.87\ \text{V} \quad \text{and} \quad R_{Th} = \frac{R_{L1} R_{L2} \left(v_1 - v_2\right)}{R_{L1} v_2 - R_{L2} v_1} = 1.03\ \Omega.$$

So for each battery, $V \approx 1.4$ V and $R \approx 0.5\ \Omega$. Batteries have resistance!

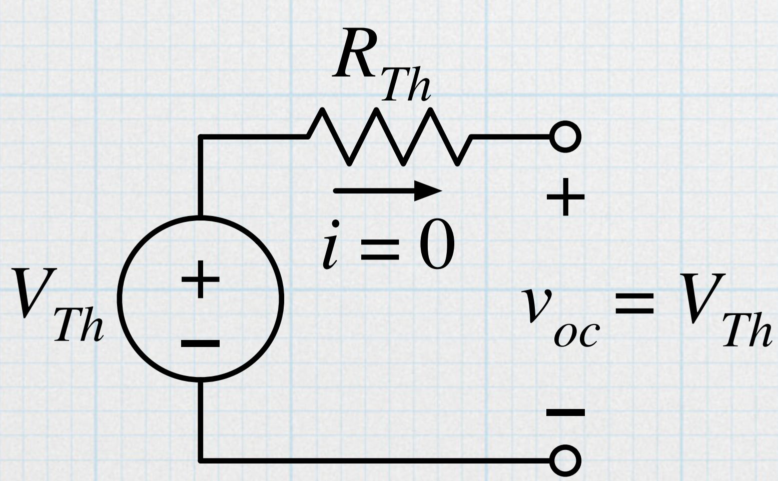
[1] I frequently use this battery setup to power electronics circuits.

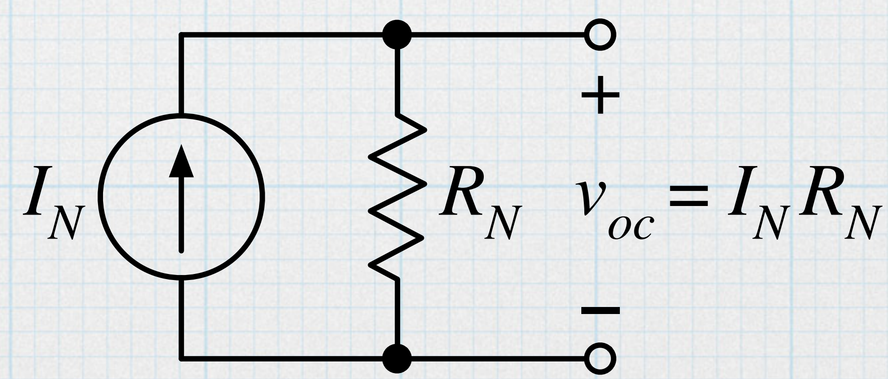
open-circuit voltage / short-circuit current

Using two random loads to determine the values in the equivalent circuit seems pretty simple. There may be some numerical issues if we use load resistors that are much bigger or much smaller than R_{Th}, but the method should work in general.

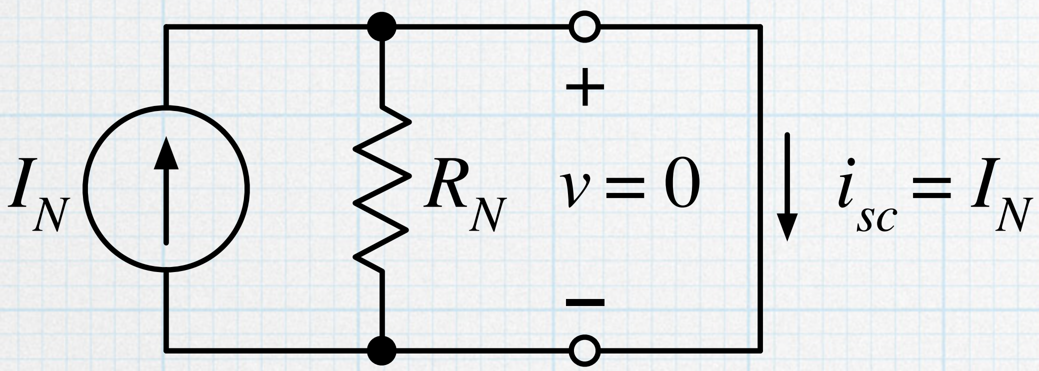
However, by being a bit clever about the choice of R_{L1} and R_{L2}, we can make the method even simpler. What if we chose R_{L1} to be an open circuit? (i.e. $R_{L1} \rightarrow \infty$) From the Thevenin circuit, we see that there will be no current due to the open circuit. Hence, there is no voltage drop across R_{Th} and the open-circuit voltage v_{oc} must be equal to V_{Th}! Boom — one measurement, one piece of the Thevenin. 🤯 On the other hand, using an open circuit with the Norton equivalent shows that $v_{oc} = I_N R_N$, so $V_{Th} = I_N R_N$.

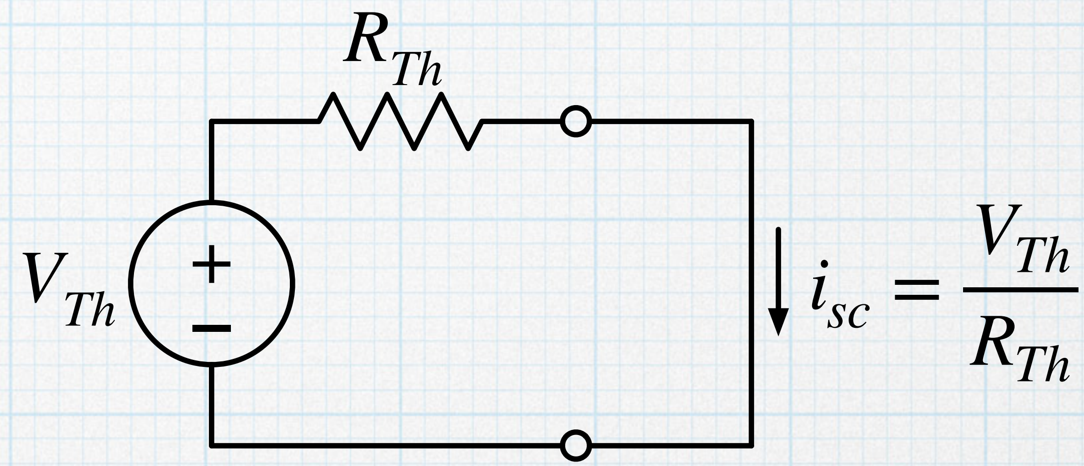
But we are not finished.





We still need a second measurement. What if we choose R_{L2} to be a short-circuit ($R_{L2} = 0$)? In the Norton circuit, we see that the short circuit at the load effectively shorts out R_N (no voltage across means no current through), and all of the source current must flow through the short, $i_{sc} = I_N$!





In shorting the output of the Thevenin circuit, we see that

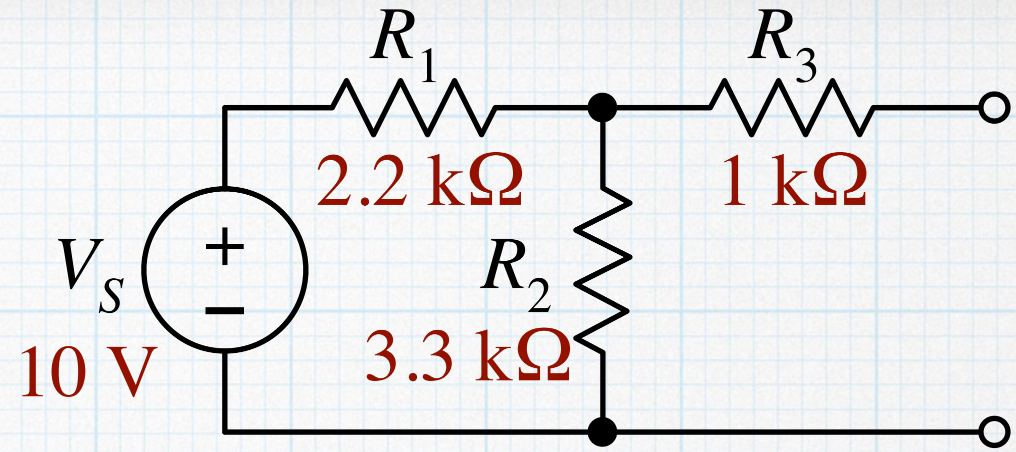
$$i_{sc} = \frac{V_{Th}}{R_{Th}}, \text{ and so } I_N = \frac{V_{Th}}{R_{Th}}.$$

Two simple measurements or calculations tell use every thing we need:

1. From the open-circuit voltage, we find V_{Th}.
2. From the short-circuit current, we find I_N.
3. Then we can calculate $R_{Th} = R_N = V_{Th} / I_N = v_{oc} / i_{sc}$.

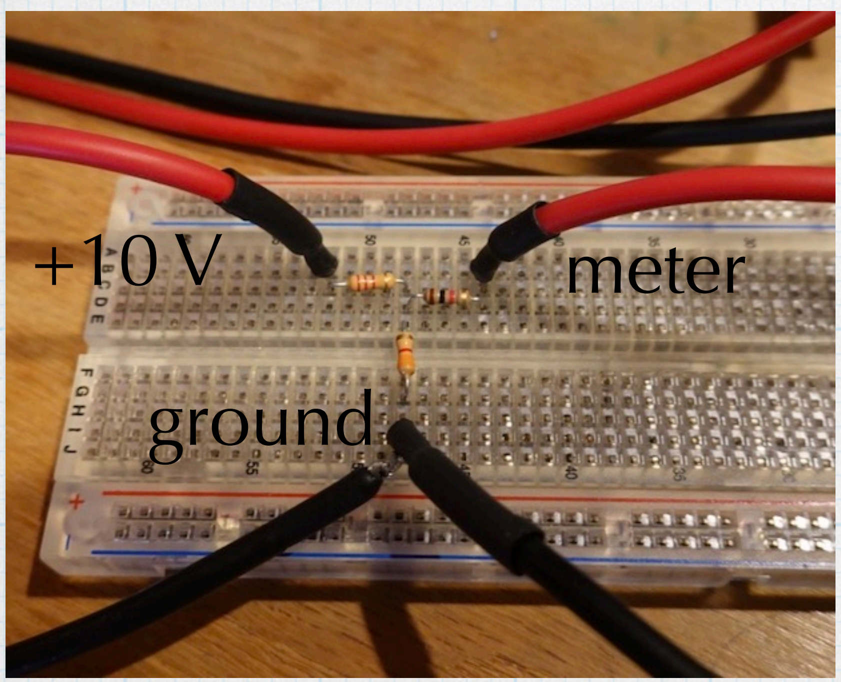
Example 3a

Let's try the v_{oc} - i_{sc} technique on a simple example. The circuit at right is a nice one.

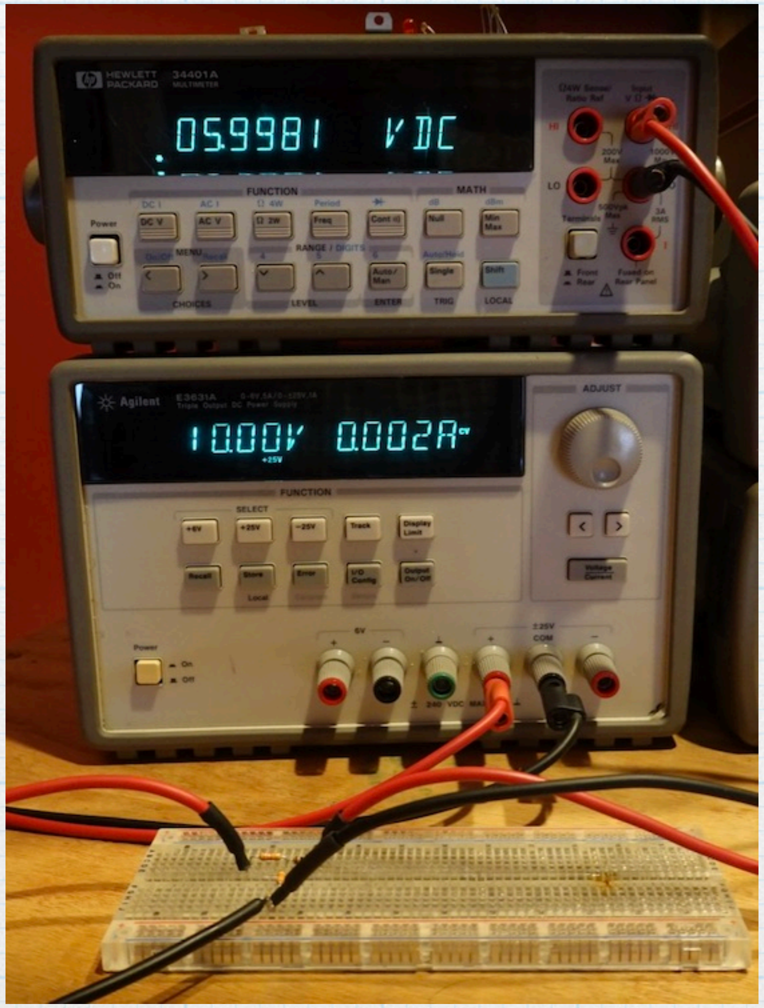


First, do the measurement in the lab. It's easy — just build the circuit on a breadboard with the 10-V supply for V_S. Use the multimeter at the port to first measure v_{oc} and then switch over to measure i_{sc}.

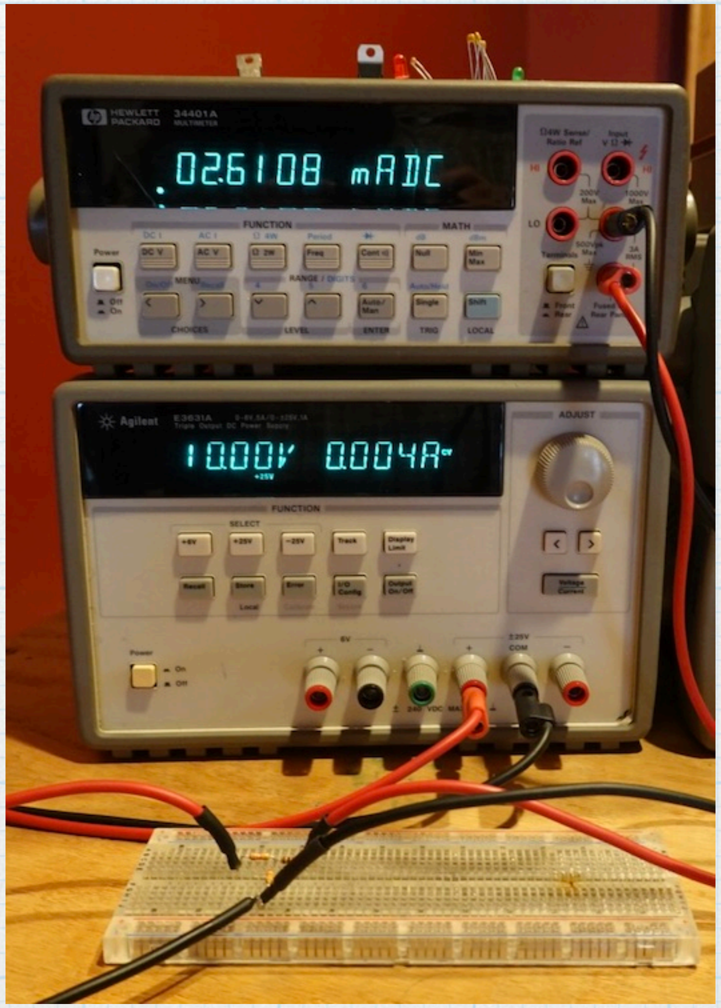
The circuit



$v_{oc} = 6\text{ V} = V_{TH}$



$i_{sc} = 2.61\text{ mA} = I_N$

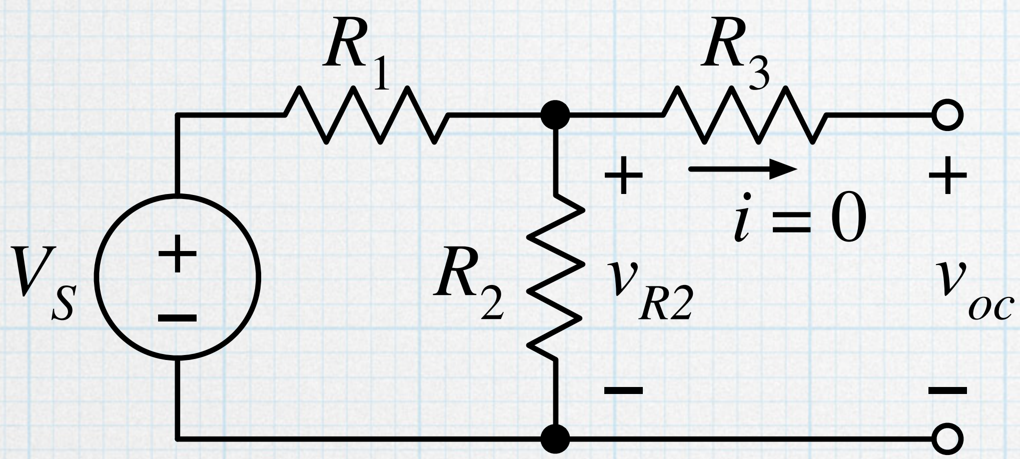


$$R_{Th} = \frac{6\text{ V}}{2.61\text{ mA}} = 2.30\text{ k}\Omega$$

Example 3b

Now let's do the same circuit with pencil and paper.

First v_{oc}. There is no current in the dangling R_3 due to the open circuit, and so there is no voltage across it. Consequently, $v_{oc} = v_{R2}$.



$$v_{oc} = v_{R2} = \frac{R_2}{R_1 + R_2} V_S$$

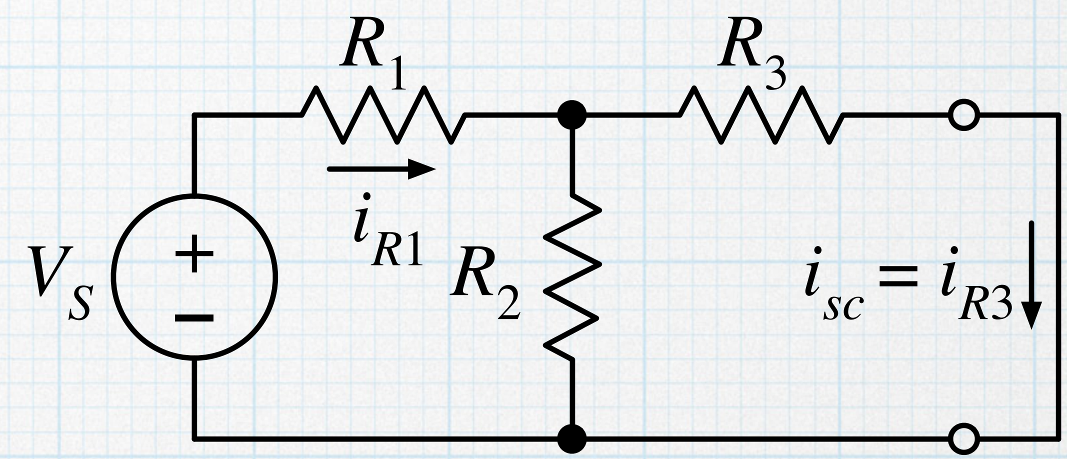
$$= \frac{3.3\,\text{k}\Omega}{2.2\,\text{k}\Omega + 3.3\,\text{k}\Omega}(10\,\text{V}) = 6\,\text{V}$$

$V_{Th} = v_{oc} = 6V \quad I_N = i_{sc} = 2.59\,\text{mA}$

$R_{TH} = v_{oc}/i_{sc} = 2.32\,\text{k}\Omega$

Consistent with lab — with slight difference due to resistor tolerance.

Next i_{sc}. Clearly $i_{sc} = i_{R3}$. First find i_{R1}. Then use a current divider to find i_{R3}.



$$i_{R1} = \frac{V_S}{R_{123}} = \frac{V_S}{R_1 + R_2 \| R_3}$$

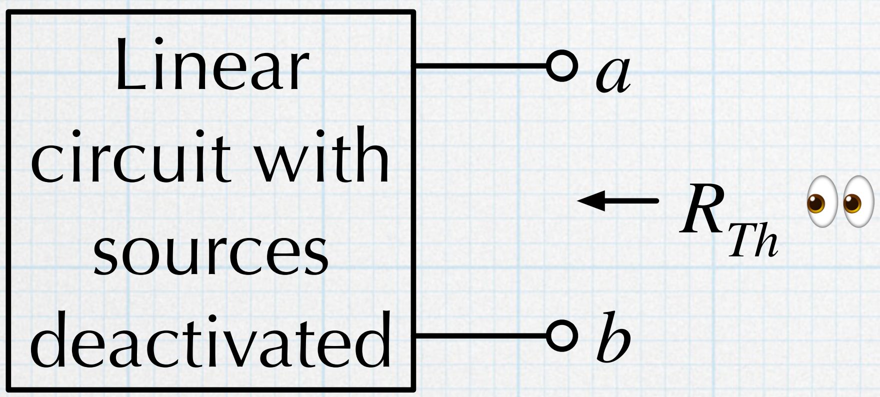
$$= \frac{10\,\text{V}}{1\,\text{k}\Omega + 2.2\,\text{k}\Omega \| 3.3\,\text{k}\Omega} = 3.37\,\text{mA}$$

$$i_{sc} = i_{R3} = \frac{\frac{1}{R_3}}{\frac{1}{R_2} + \frac{1}{R_3}} \cdot i_{R1}$$

$$= \frac{\frac{1}{3.3\,\text{k}\Omega}}{\frac{1}{2.2\,\text{k}\Omega} + \frac{1}{3.3\,\text{k}\Omega}}(3.37\,\text{mA}) = 2.59\,\text{mA}$$

Alternative method for R_{Th}

In some circumstances, we can use a short-cut for finding the Thevenin resistance. If the original circuit has only independent sources, we can find the equivalent resistance by deactivating all the sources, and then finding the equivalent resistance looking in at the port.



This works because the equivalent resistance derives from the network of resistors in the original circuit. If we remove the sources (deactivate them), we are left with just resistors, allowing us to find the equivalent resistance using the resistor reduction techniques learned earlier.

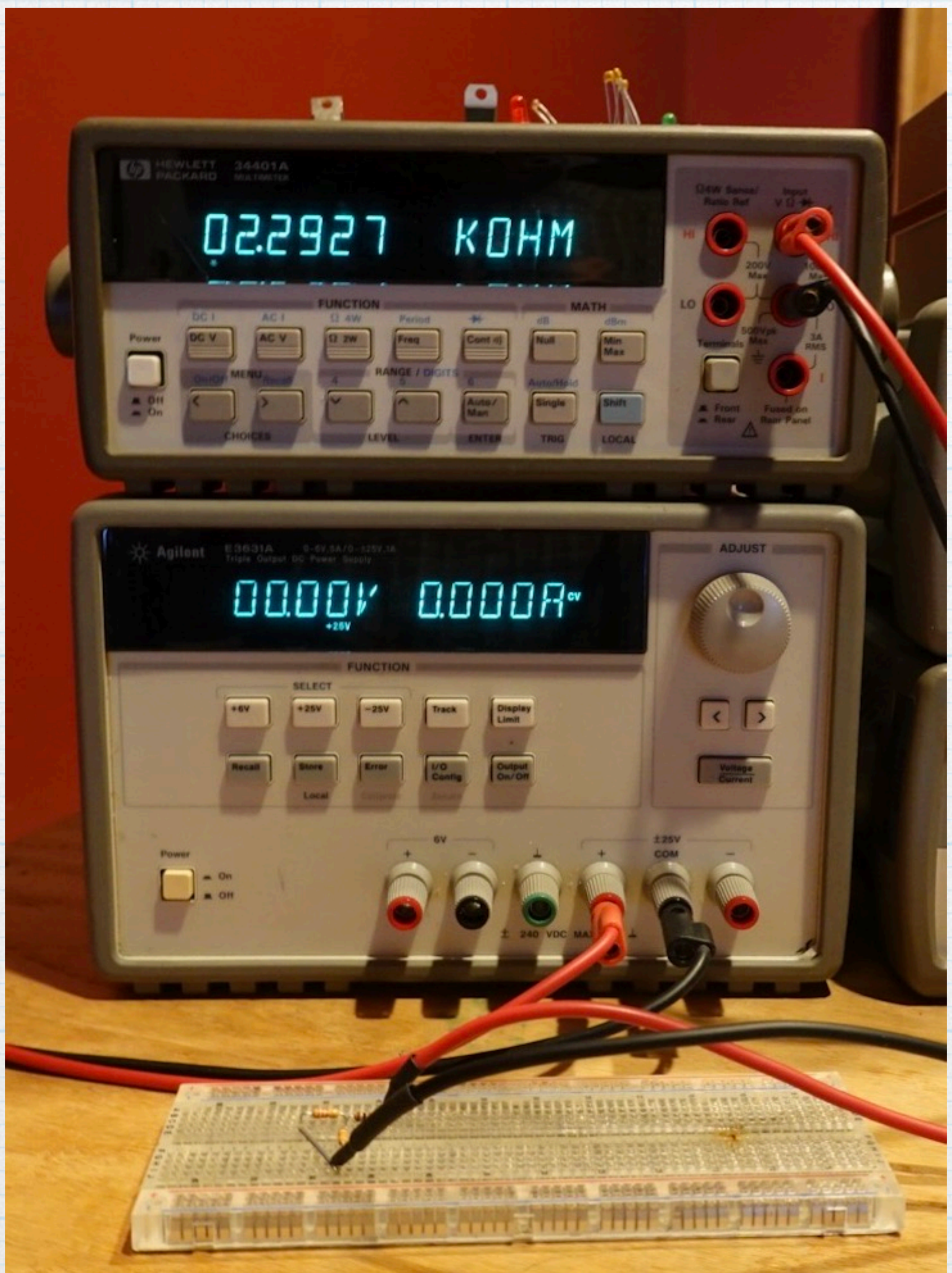
In a circuit having only independent sources, finding R_{Th} directly using this short cut might be easier than v_{oc} and then i_{sc}. Using this short cut, we could find v_{oc} and R_{Th} directly (and then calculate I_N) or find i_{sc} and R_{Th} directly (and then calculate V_{Th}).

But to re-emphasize, this short cut will not work if there are dependent sources in the circuit. Dependent sources cannot be deactivated. (Just like when doing superposition.) In that case, you *must* find v_{oc} and i_{sc} and then calculate the correct R_{Th}.

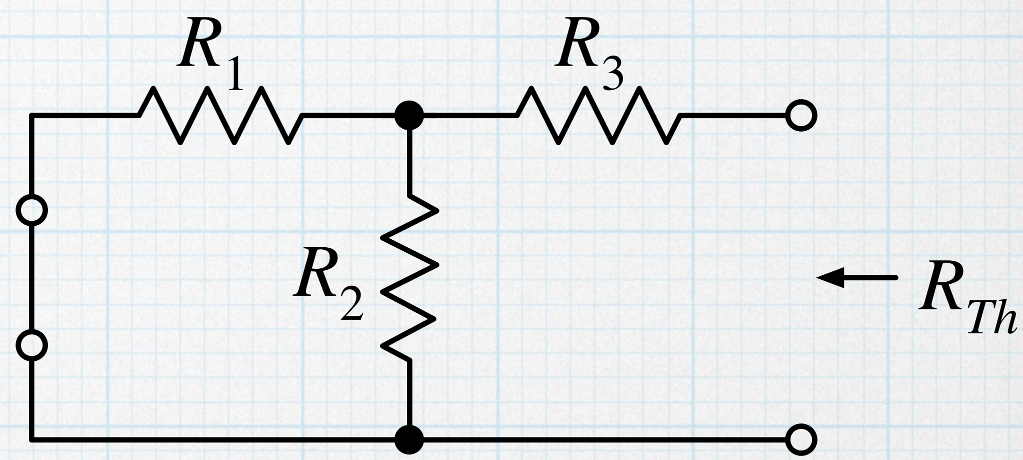
Example 3c

Use the short-cut method to find R_{Th} directly.

In lab, with the source removed and replaced with a short, use an ohm-meter at the port to find R_{Th}.



Or with pencil and paper.



$$R_{Th} = R_3 + R_1 \| R_2$$

$$= 1\,\text{k}\Omega + (2.2\,\text{k}\Omega)\|(3.3\,\text{k}\Omega)$$

$$= 2.32\,\text{k}\Omega$$

Again, the difference is attributable to resistor tolerances.

Cautions when measuring Thevenin in lab

The previous example implies that measuring equivalent circuits in the lab is straight-forward, but that is only half true. Finding open-circuit voltage is easy — just use the voltmeter at the open port. However, we must be careful if trying to directly measure short-circuit current or resistance.

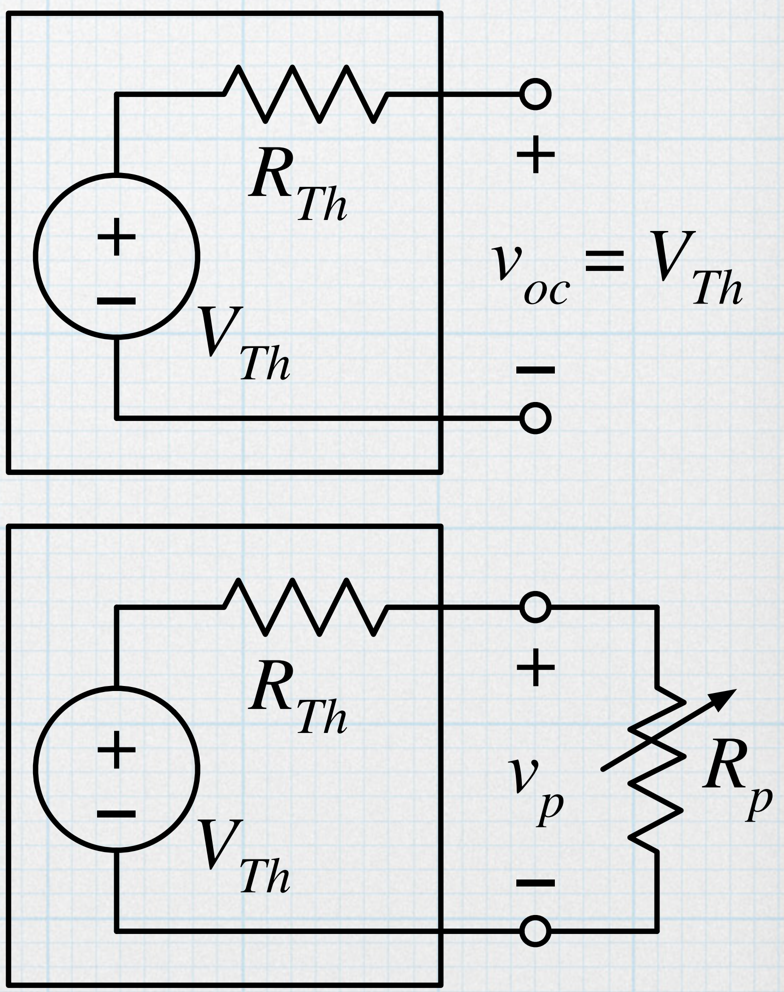
1. Directly shorting the output port of a circuit is usually not a good idea. Short circuits can result in high currents. This may cause the device to enter a current-limiting condition, if it is designed with protection. (Like the lab power supplies.) If the ammeter is used as the short circuit, the high current may cause the meter fuse to blow. In the worst case, the high current may cause damage in the circuit being tested.

2. Even if nothing burns out, an ammeter is not a true short circuit. (Look back at the DMM lab exercises.) In circuits with "small-ish" resistors, the non-zero meter resistance may affect the measurement.

3. When measuring a black box — where the details of the internal circuit are unknown — we cannot assume that there are no dependent sources. In that case, the "short-cut" method of using the ohmmeter to measure resistance directly is not reliable.

The half-voltage method for measuring R_{Th}

Since measuring equivalent resistance using short circuits or short cuts can be tricky, we are back to the original idea of attaching non-zero loads to the output, and then working backwards to find R_{Th}. We can still find V_{Th} easily by directly measuring v_{oc}. Then, we need only one resistance measurement. With a bit of cleverness, we can complete the measurement with no extra calculations. The trick is to use a potentiometer.

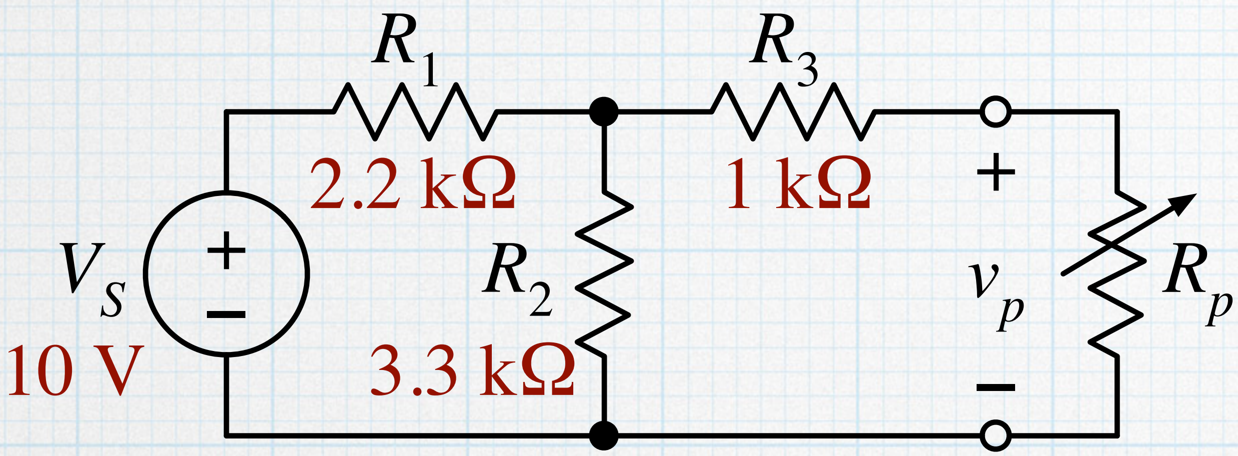


First, use the voltmeter to measure the open-circuit voltage. Then attach a potentiometer to the port. The nominal value of the potentiometer should be bigger than the expected equivalent resistance, but not too much bigger. Then, while measuring the voltage across the potentiometer with the voltmeter, adjust the resistance until the voltage is exactly half of the open-circuit voltage. At that point, $R_p = R_{Th}$. Remove the potentiometer and measure its resistance.

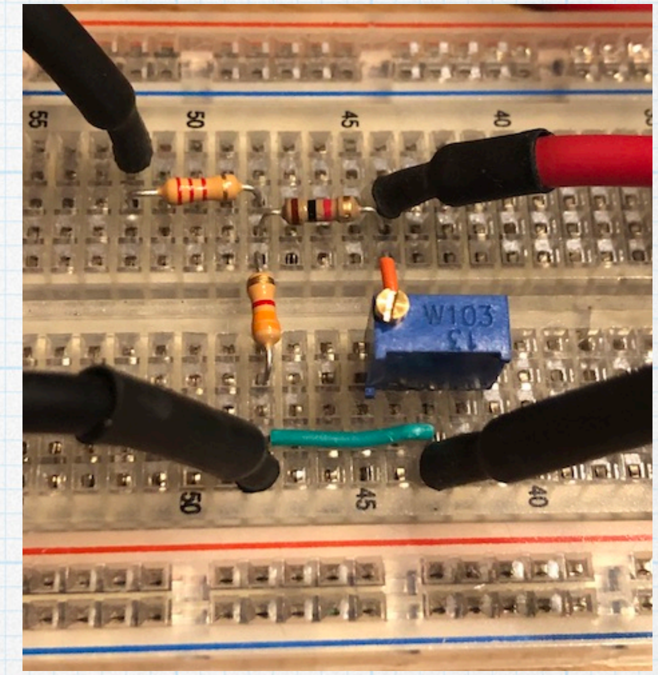
$$v_p = \frac{v_{oc}}{2} \text{ when } R_p = R_{Th}.$$

Example 3d

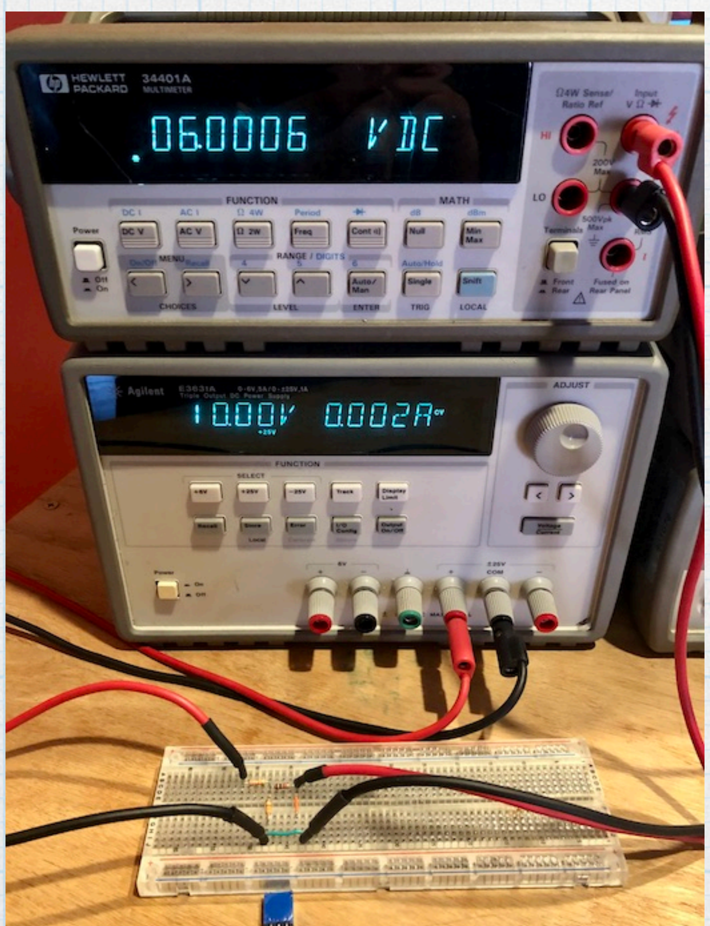
Use the “half-voltage” method to find the Thevenin resistance.



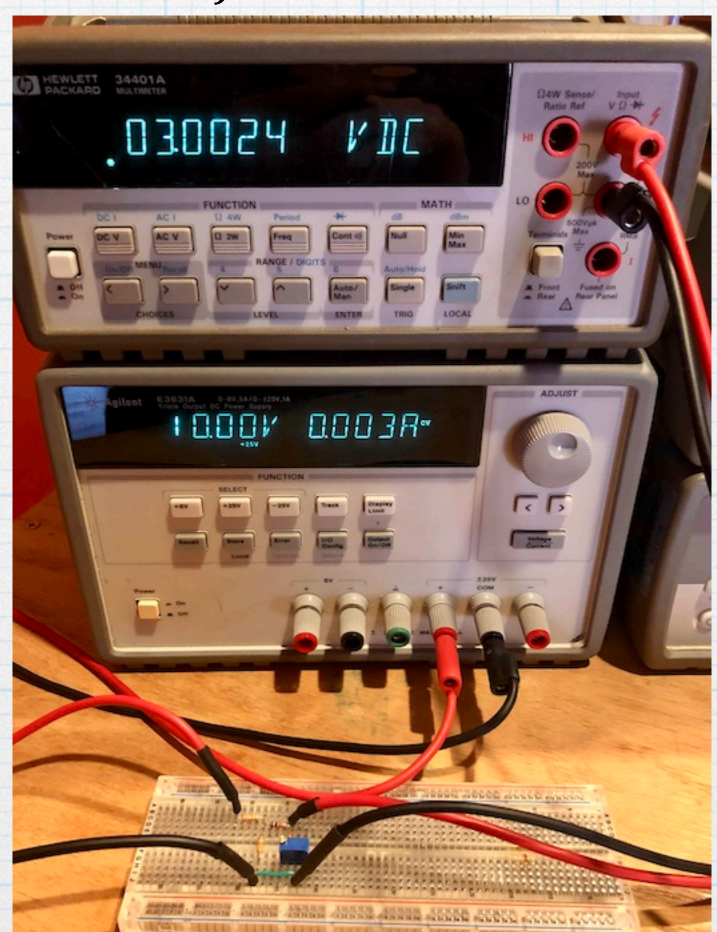
Circuit with 10-turn, 10-kΩ potentiometer as the load.



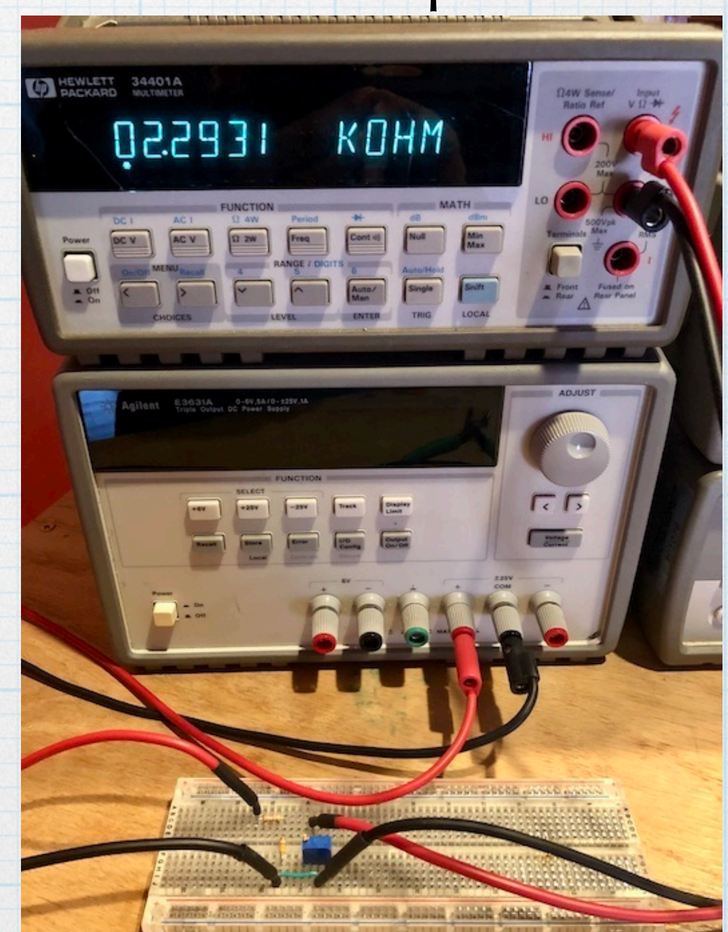
Pot removed. Measure v_{oc}.



Insert pot. Adjust to $v_{oc}/2$.



Disconnect. Measure R_{Th} from the pot.



Measuring Thevenin / Norton equivalents

Suggested approach

1. Use a voltmeter to measure *open-circuit* voltage at the port: $v_{oc} = V_{Th}$.
2. Attach a resistor to the port — small enough to ensure good current flow. (The current should be an appreciable fraction of expected I_N.) Measure the port voltage and use the result along with v_{oc} to calculate R_{Th}.
3. Calculate $I_N = v_{oc} / R_{Th}$.

Alternatives (But use caution.)

a. Measure R_{Th} by using a potentiometer to implement the "half-voltage method". Should work, but watch for high currents.

b. Use an ammeter as the short circuit. Beware of high currents that can damage the circuit or the meter. If R_{Th} is small, the measurement may limited by meter resistance.

c. Deactivate all independent sources. Use an ohmmeter to measure R_{Th} directly. May be incorrect if there are dependent sources in the circuit, which is often the case with electronics.

Calculating Thevenin / Norton equivalents

Approach that always works

1. Using whatever analysis techniques you prefer, calculate the open-circuit voltage at the port: $v_{oc} = V_{Th}$.
2. Connect a short circuit across the output. Using whatever techniques are appropriate, calculate the short-circuit current: $i_{sc} = I_N$.
3. Calculate $R_{Th} = R_N = v_{oc} / i_{sc}$.

Short-cut method for resistance.
(Works only if there are no dependent sources.)

a. Deactivate all independent sources, and then calculate the equivalent resistance seen looking in at the port: $R_{eq} = R_{Th} = R_N$.

b. Calculate *either* $v_{oc} = V_{Th}$ or $i_{sc} = I_N$ (your choice) as in described in steps 1 or 2 above.

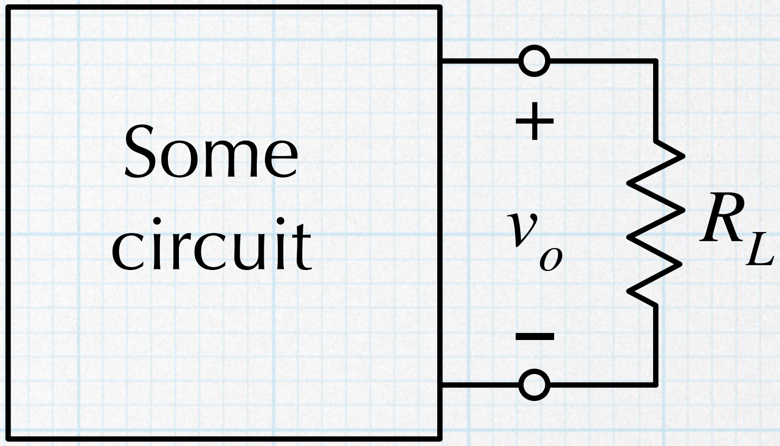
c. Calculate the other source using the resistance from part a: $V_{Th} = R_{eq} \cdot I_N$.

When doing Thevenin/Norton calculations...

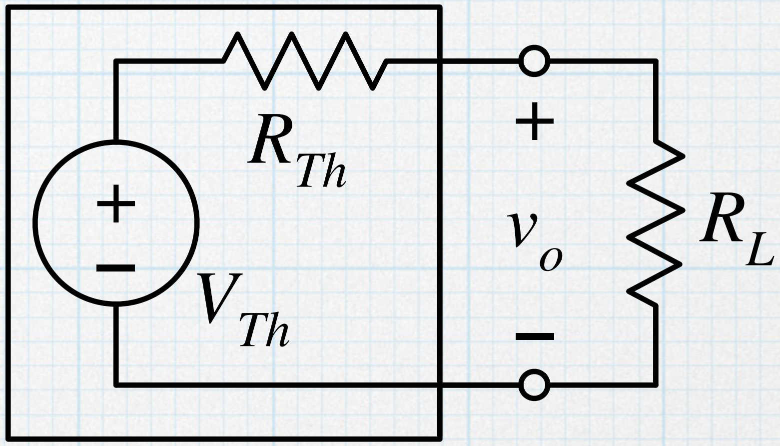
There are several examples illustrating calculation techniques in the companion set of notes, "Thevenin/Norton Examples". Specific things to note:

1. All the techniques learned earlier (including SPICE) are fair game for calculating V_{Th}, I_N, or R_{Th}. Use whatever seems appropriate for the circuit.
2. The v_{oc} and i_{sc} calculations are done on different circuits. Do not try to use results from the v_{oc} calculation in the i_{sc} circuit. This is a common mistake. Shorting the port changes the circuit — at the very least, there is one less node.
3. Do not use the short-cut method for resistance if there are dependent sources.
4. When doing v_{oc} calculations, do not be fooled by "dangling" resistors. A dangling resistor with no current has no voltage across it. It can be treated as if it were just wire in the v_{oc} calculation.
5. When the port is shorted, shunt resistors will be shorted out. A shorted resistor has no voltage and hence no current. It can be ignored in the i_{sc} calculation.
6. Finding the current in a wire by itself might seem weird — we never calculated the current in a wire in any previous examples. However, KCL still applies. Look at a node connected to the wire and find i_{sc} by summing up the other currents flowing into the node.

Maximum power transfer



Max power?



Now that we can model any linear circuit using a simple Thevenin (or Norton) equivalent, we can answer another important question that frequently arises in circuit design: Given a circuit, what value of load resistance results in maximum power delivered to the load?

If we know the Thevenin equivalent of the circuit, the answer turns out to be quite simple.

With the load attached, the output voltage is

$$v_o = \frac{R_L}{R_{Th} + R_L} V_{Th}$$

and the corresponding power delivered to the load is

$$P_L = \frac{V_{Th}^2 R_L}{\left(R_{Th} + R_L\right)^2}$$

If we view the power as a function of R_L, we can find the maximum by using the usual calculus trick of taking the derivative and setting it equal to zero.

$$\frac{dP_L}{dR_L} = \frac{V_{Th}^2}{\left(R_{Th} + R_L\right)^2} - 2\frac{R_L V_{Th}^2}{\left(R_{Th} + R_L\right)^3} = 0$$

Multiplying both sides by $\left(R_{Th} + R_L\right)^3$ and dividing by V_{Th}^2, leaves

$$R_{Th} + R_L - 2R_L = 0.$$

Finishing:

$R_L = R_{Th}$ for maximum power delivered to the load.

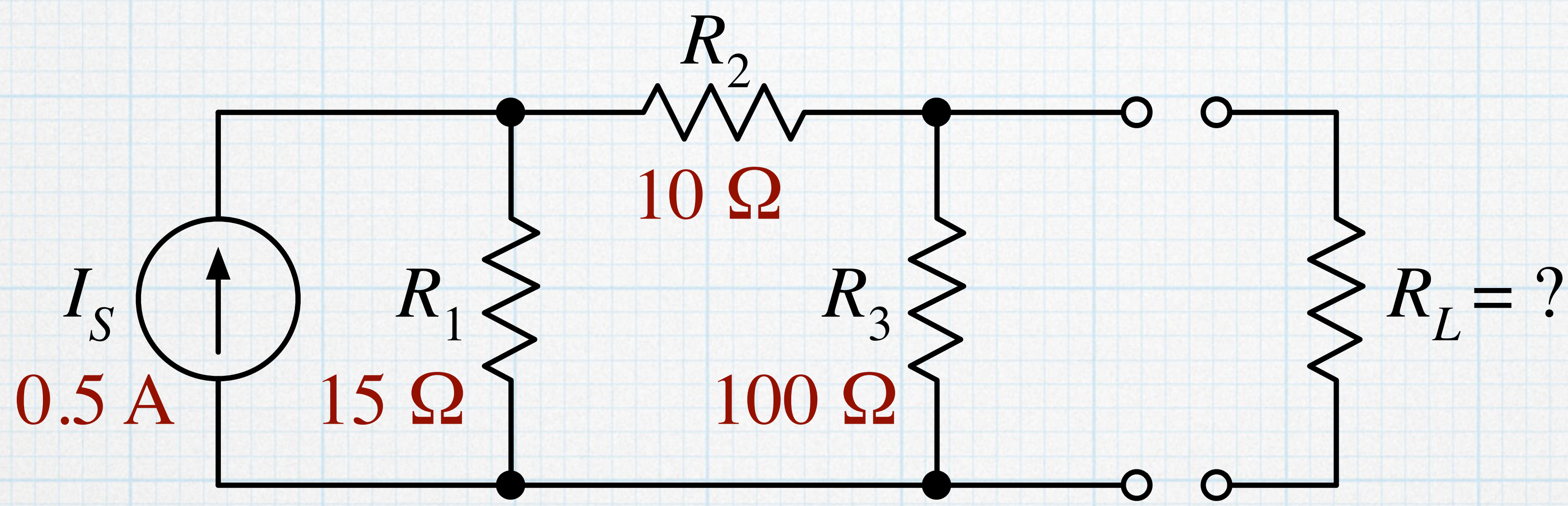
With $R_L = R_{Th}$, $v_{RL} = V_{Th}/2$ (or $i_{RL} = I_N/2$) — only half the voltage (or current) appears at the output port. This may seem surprising, but recall that we are maximizing for power, not voltage or current.

Also, half of the power generated by the equivalent source is dissipated in the equivalent resistance. Again, this may seem surprising, but it is a consequence of maximizing power delivered to the load.

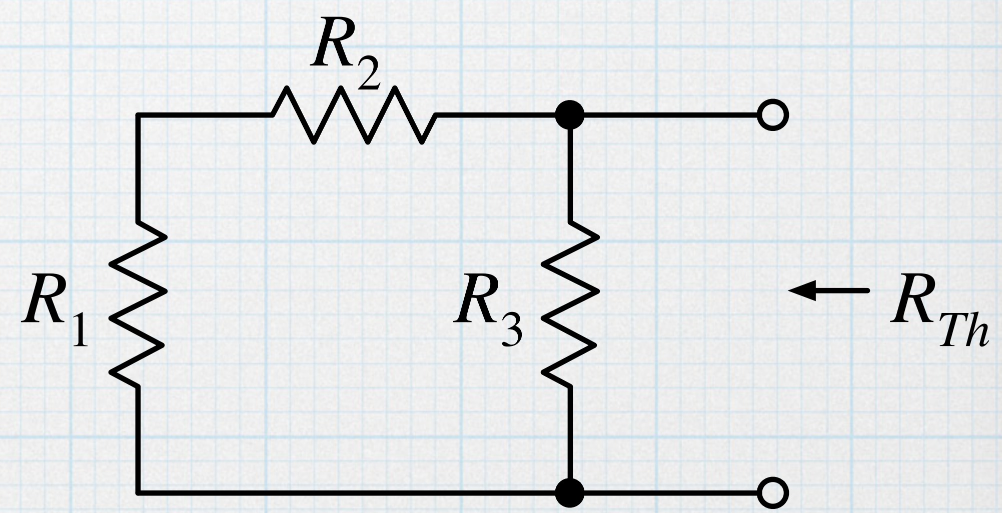
Refer back to the first example and the tables on pages 6 & 7 — the delivered power clearly peaks when $R_L = R_{Th}$.

Example 4

For the circuit below, find the value of load resistance so that maximum power is delivered to the resistor. What is that maximum power ?



This is a Thevenin/Norton equivalent problem in disguise. Obviously, we need to find an equivalent circuit. We can start by finding the resistance. Since there is only a single independent source, we can use the short-cut method. After deactivating I_S:



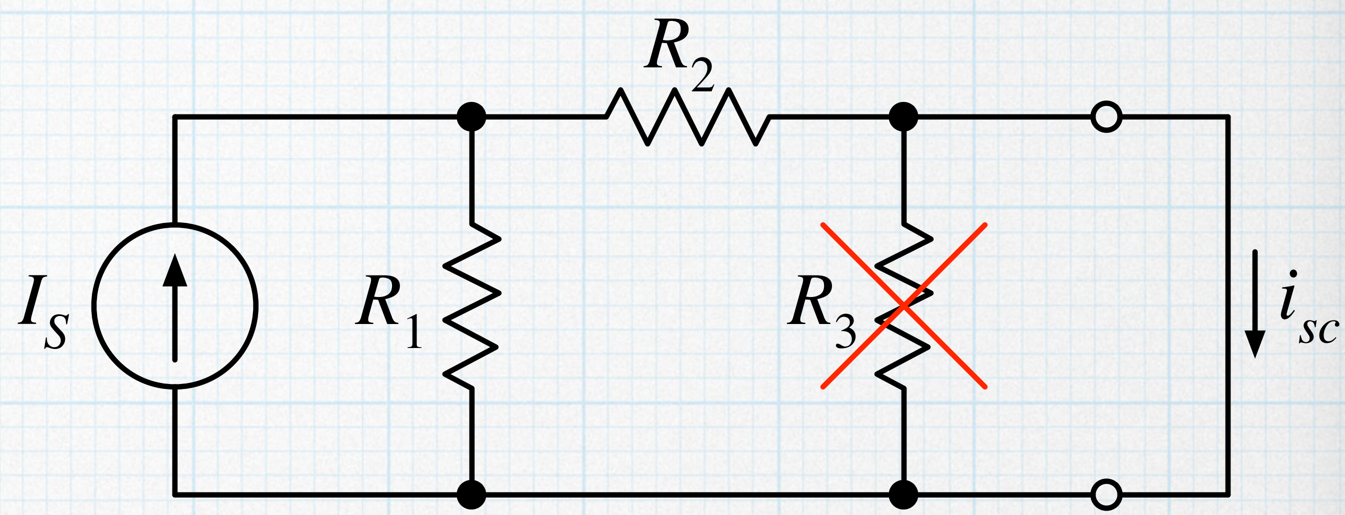
$$R_{eq} = R_{Th} = R_N = R_3 \| (R_1 + R_2) = 20\,\Omega$$

So R_L should be 20 Ω.

Example 4 (cont.)

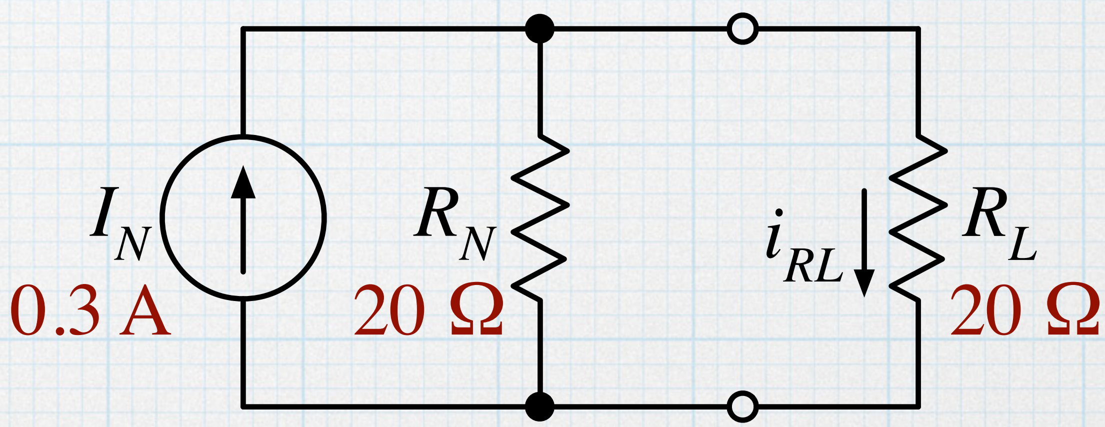
To calculate the actual power, we need either V_{TH} or I_N. Let's use I_N.

Shorting the output — which shorts R_3 — we can see that $i_{sc} = i_{R2}$. Using a current divider:



$$i_{sc} = i_{R2} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} I_S = 0.3\text{ A}.$$

So $I_N = 0.3$ A.



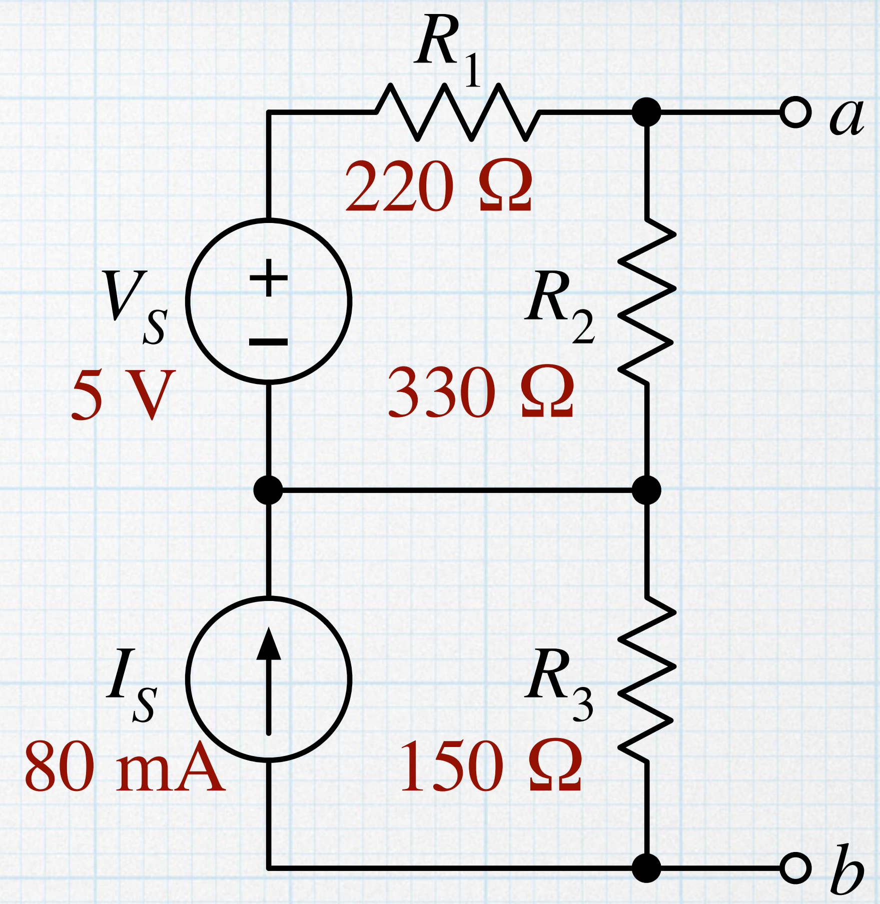
Finally, the load power is

$$P_L = i_{RL}^2 R_L = \left(\frac{I_N}{2}\right)^2 R_L = 0.45\text{ W}$$

Something to ponder: Suppose the question is reversed — given an R_L of 20 Ω and $I_S = 0.5$ A, what values of R_1, R_2, and R_3 should we choose in the circuit to deliver maximum power to R_L? (Beware — trick question!)

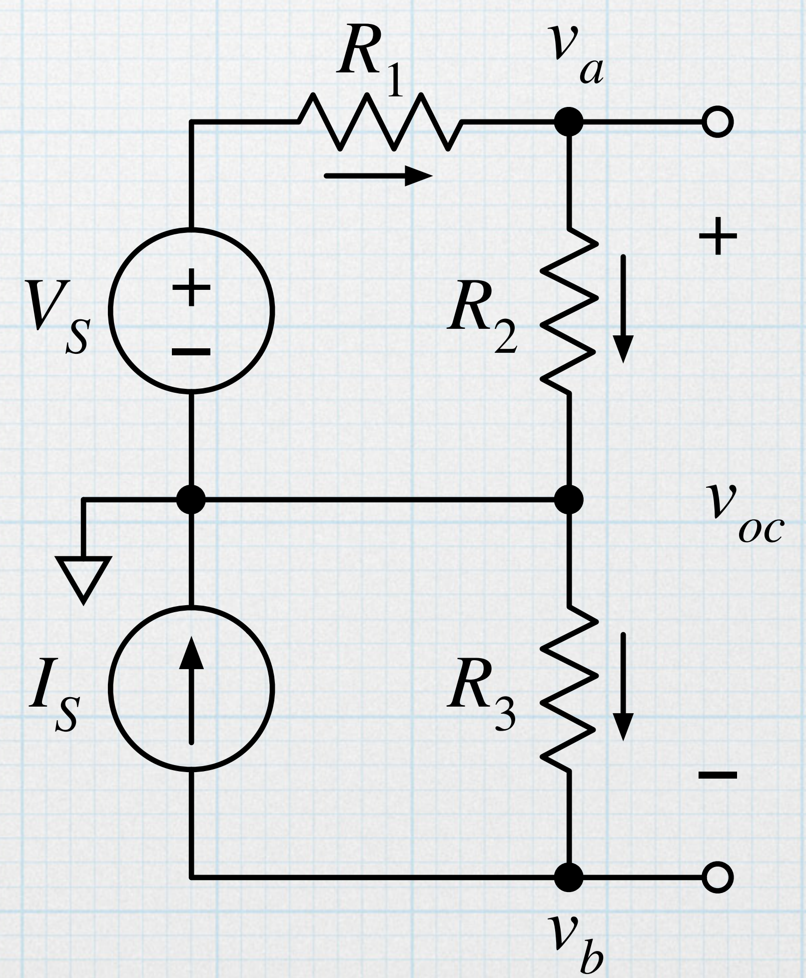
Example 5

Find the Thevenin and Norton equivalents for the circuit shown at right with respect to the port defined by *a* and *b*.



This circuit looks rather odd, but we can sort it out easily by defining a ground and writing node-voltage equations at *a* and *b*.

$$\frac{V_S - v_a}{R_1} = \frac{v_a}{R_2} \quad \text{and} \quad \frac{-v_b}{R_3} = I_S.$$



The two nodes are independent. So finding the voltages is quite easy.

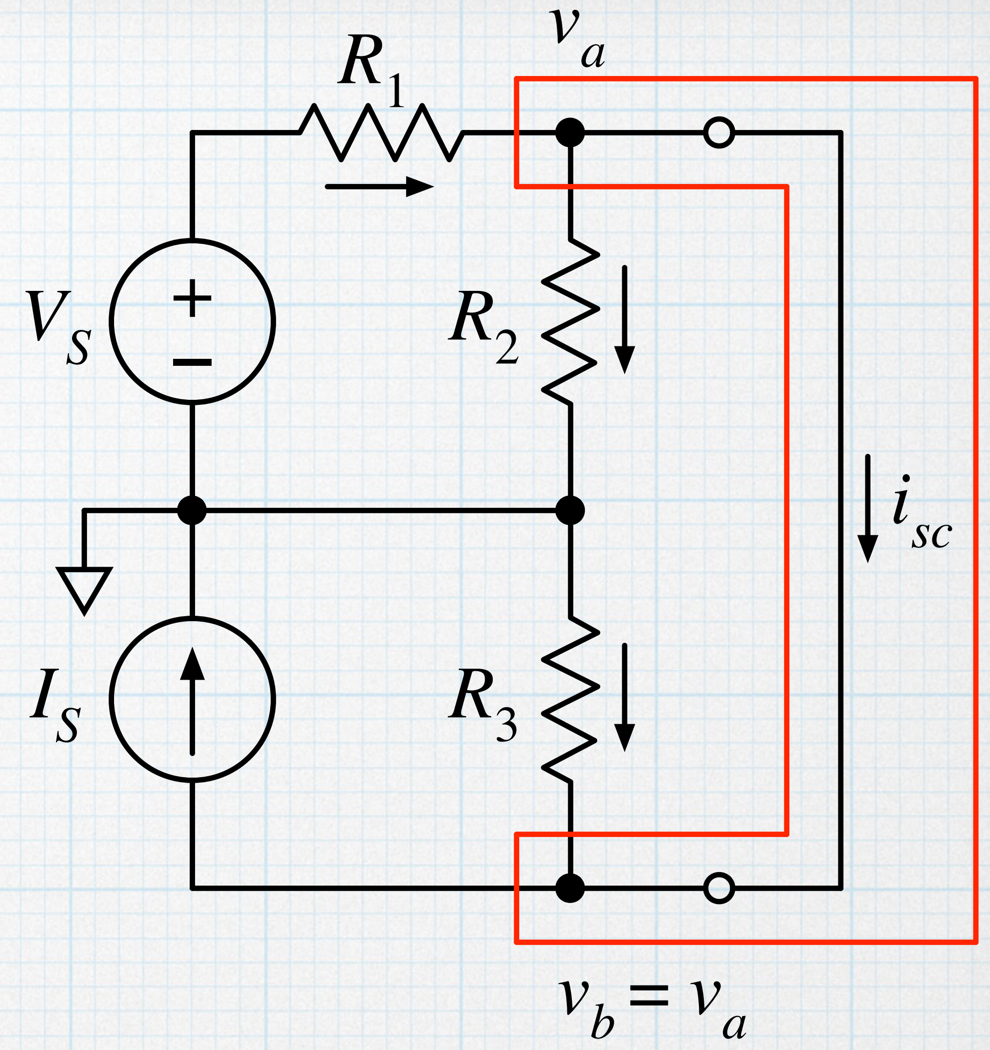
$$v_a = \frac{R_2}{R_1 + R_2} V_S = 3\text{ V and}$$

$$v_b = -R_3 \cdot I_S = -12\text{ V}.$$

Finally, $V_{Th} = v_{oc} = v_a - v_b = 15$ V.

Example 5 (cont.)

Find the short-circuit current. This looks even stranger. First, R_2 and R_3 are *not* shorted out because the ground connection allows for an alternate path for current to flow. Also, because of the short, $v_b = v_a$, so there is only one unknown node. (As outlined by the red box.)



Finally, $i_{sc} = i_{R1} - i_{R2}$. (Also $i_{sc} + i_{R3} = I_S$.) Either way, we need to find the node voltage. Summing currents carefully,

$$i_{R1} + i_{R3} = i_{R2} + I_S.$$

$$\frac{V_S - v_a}{R_1} + \frac{-v_a}{R_3} = \frac{v_a}{R_2} + I_S.$$

$$v_a = \frac{V_S - R_1 I_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}}.$$

Plugging in numbers gives $v_a = -4.02$ V.

Finally, $i_{sc} = i_{R1} - i_{R2} = \dfrac{V_S - v_a}{R_1} - \dfrac{v_a}{R_2} = 53.2\,\text{mA}$

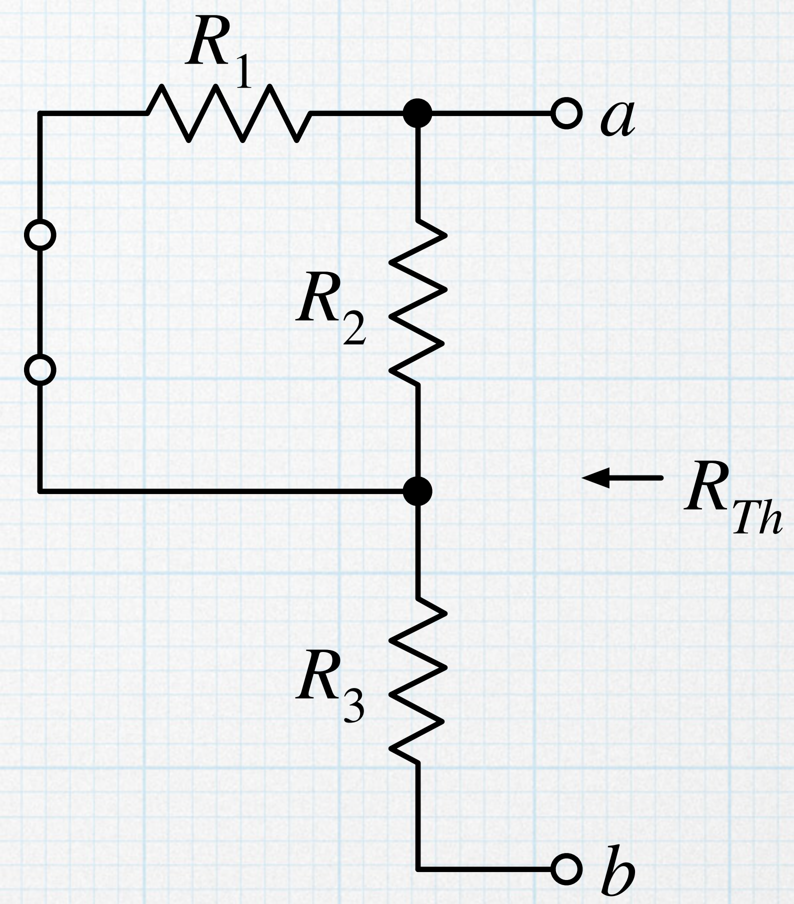
$$R_{Th} = \frac{v_{oc}}{i_{sc}}$$

$$= 282\,\Omega$$

Example 5 (cont.)

Or we might choose to use the short-cut method to find R_{Th}, since there are only independent sources in the circuit.

After deactivating the sources (I_S removed, V_S shorted), the resistor network is shown.



It is pretty easy to see the equivalent resistance between a and b:

$$R_{Th} = R_3 + R_1 \| R_2$$

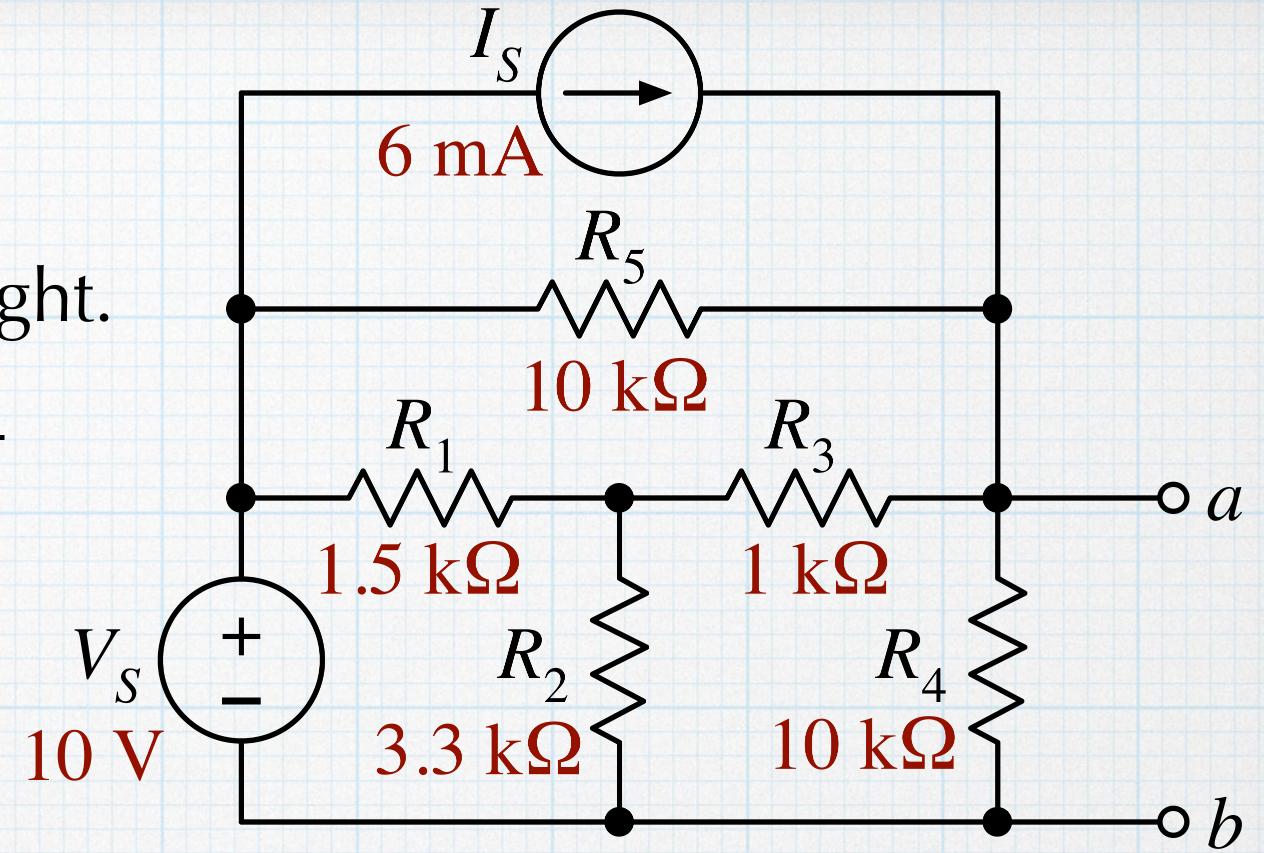
$$= 150\,\Omega + (220\,\Omega) \,\|\, (330\,\Omega)$$

$$= 282\,\Omega$$

As expected.

Example 6

Find the Thevenin and Norton equivalents for the circuit shown at right.



First, find v_{oc}. Clearly, $v_{oc} = v_{R4}$. Node-voltage analysis might be the most expedient approach. Then $v_{oc} = v_y$.

$$i_{R1} = i_{R2} + i_{R3}$$

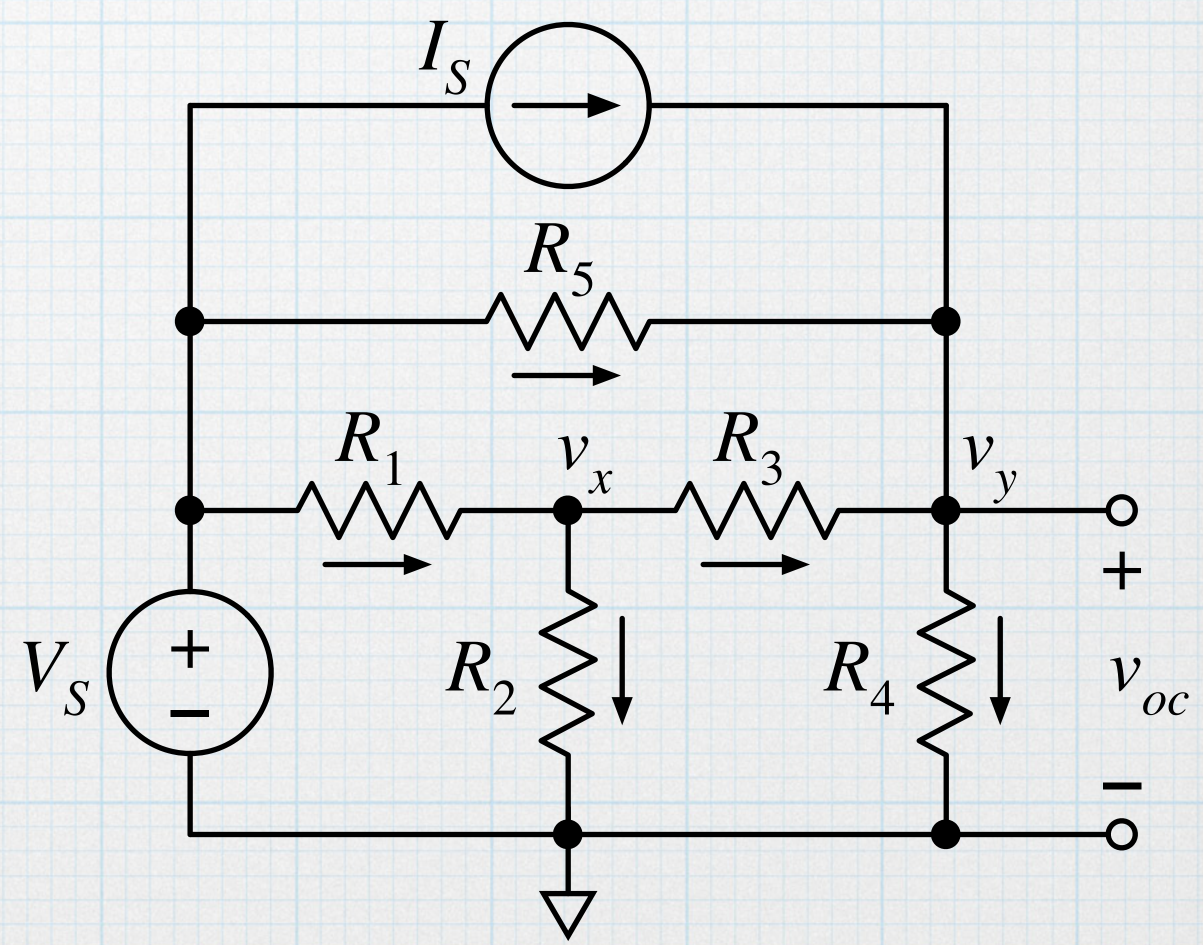
$$i_{R1} + i_{R5} + I_S = i_{R4}$$

$$\frac{V_s - v_x}{R_1} = \frac{v_x}{R_2} + \frac{v_x - v_y}{R_3}$$

$$\frac{v_x - v_y}{R_3} + \frac{V_s - v_y}{R_5} + I_S = \frac{v_y}{R_4}$$

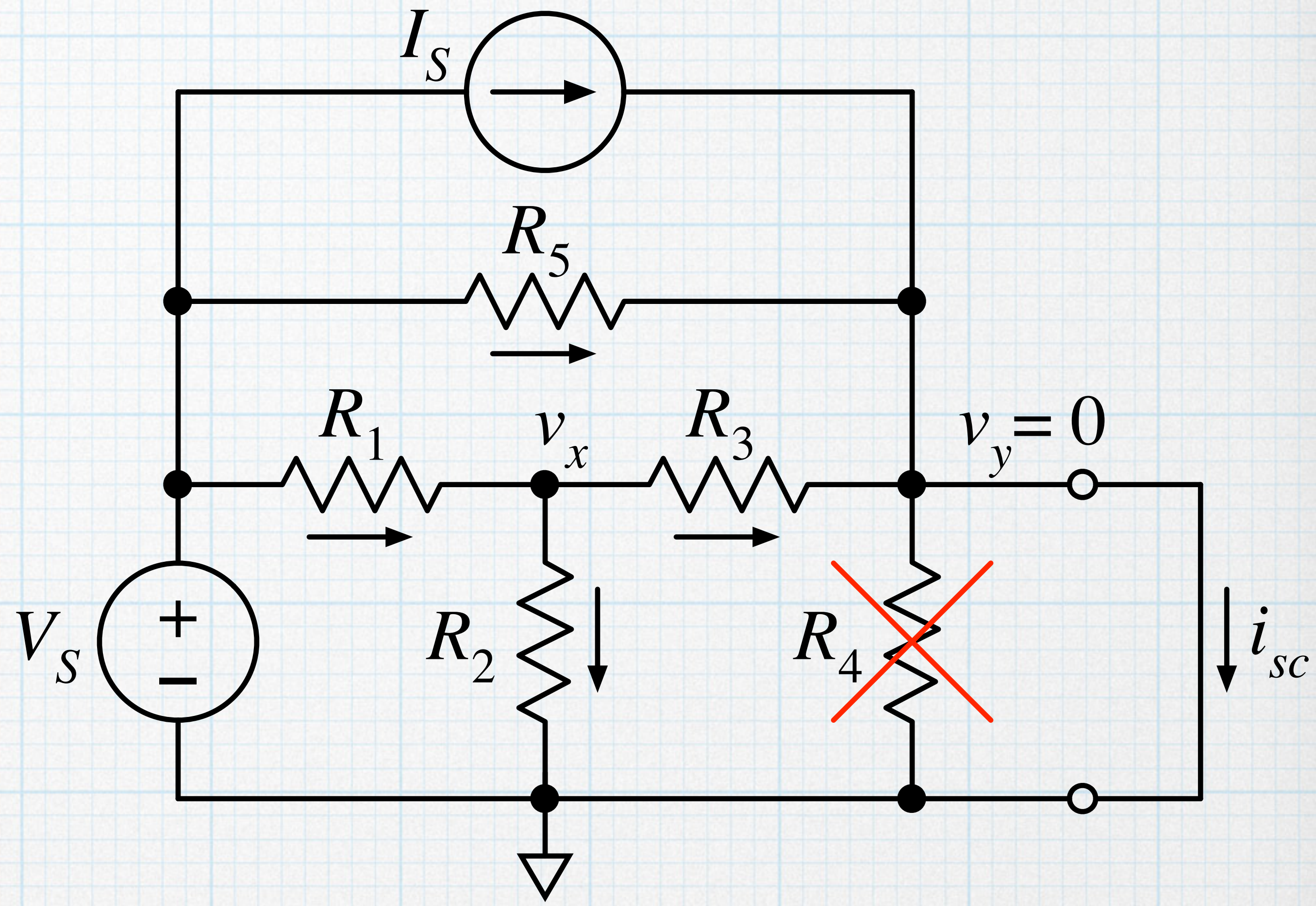
$$\left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3}\right)v_x - \frac{R_1}{R_2}v_y = V_S$$

$$-\frac{R_5}{R_3}v_x + \left(1 + \frac{R_5}{R_3} + \frac{R_5}{R_4}\right)v_y = V_S + R_5 I_S$$



Inserting values and solving gives $v_x = 11$ V and $v_y = 15$ V. So $V_{Th} = 15$ V.

Example 6 (cont.)



Next, calculate the short-circuit current. Start by noting that R_4 will be shorted out. Also, with the port shorted, $v_y = 0$.

Using KCL, we see that

$$i_{sc} = i_{R3} + i_{R5} + I_S$$

$$i_{R5} = \frac{V_S - v_y}{R_5} = \frac{V_S}{R_5} = 1\,\text{mA}$$

To find i_{R3}, we will need to determine the node voltage, v_x. Writing the NV equation:

$$\frac{V_S - v_x}{R_1} = \frac{v_x}{R_2} + \frac{v_x}{R_3}$$

$$v_x = \frac{V_S}{1 + \frac{R_2}{R_1} + \frac{R_3}{R_1}}$$

Inserting values gives $v_x = 3.385$ V.

Then:

$$i_{sc} = \frac{3.385\,\text{V}}{1\,\text{k}\Omega} + 1\,\text{mA} + 6\,\text{mA}$$

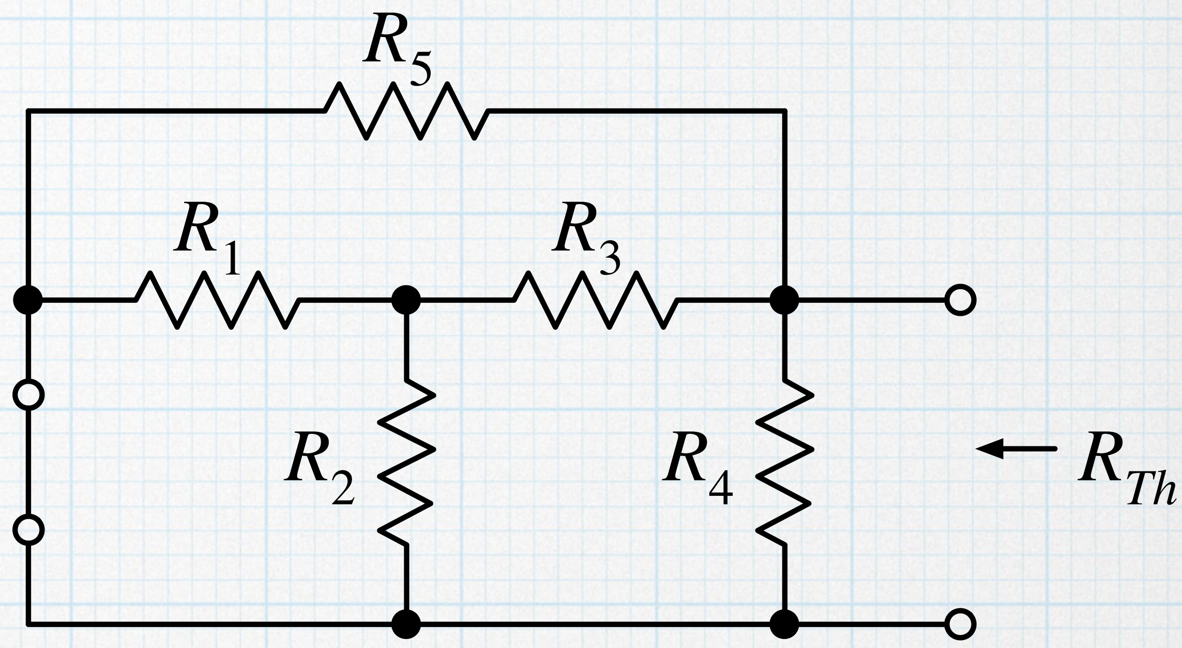
$$= 10.385\,\text{mA}$$

$$I_N = i_{sc} = 10.385\ \text{mA}$$

Finally,

$$R_{Th} = v_{oc} / i_{sc} = 1.44\ \text{k}\Omega$$

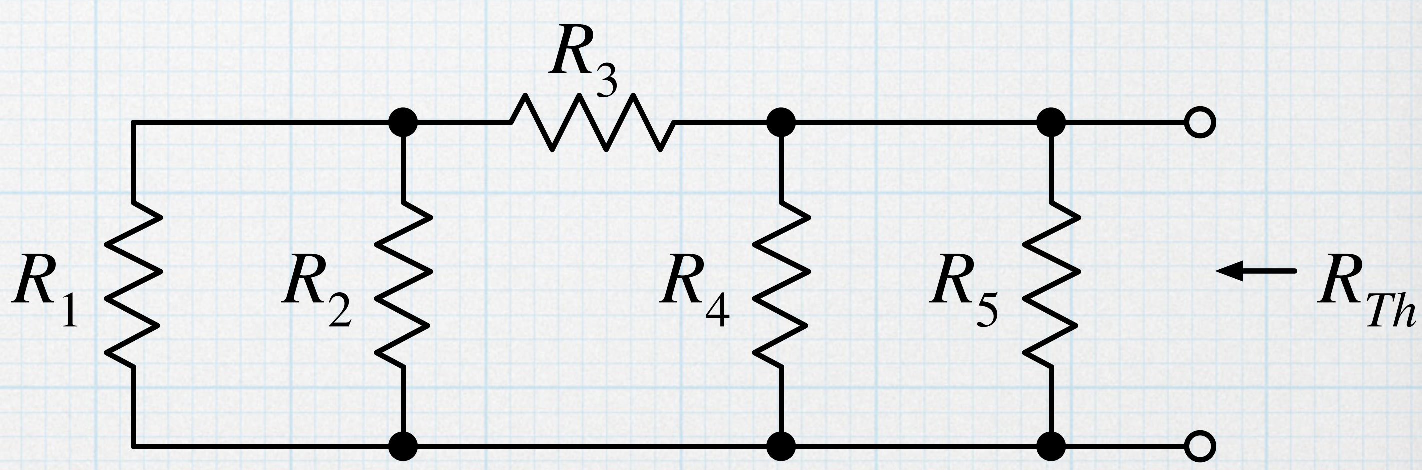
Example 6 (cont.)



Or we might choose to use the short-cut method to find R_{Th}, since there are only independent sources in the circuit.

After deactivating the sources (I_S removed, V_S shorted), the resistor network is shown above right.

Redraw.



After noting the arrangement of the resistors, we can redraw the network — it might be easier.

Then, from the point of view of the port, the resistance is

$$R_{Th} = R_5 \| R_4 \| \left(R_3 + R_1 \| R_2\right)$$

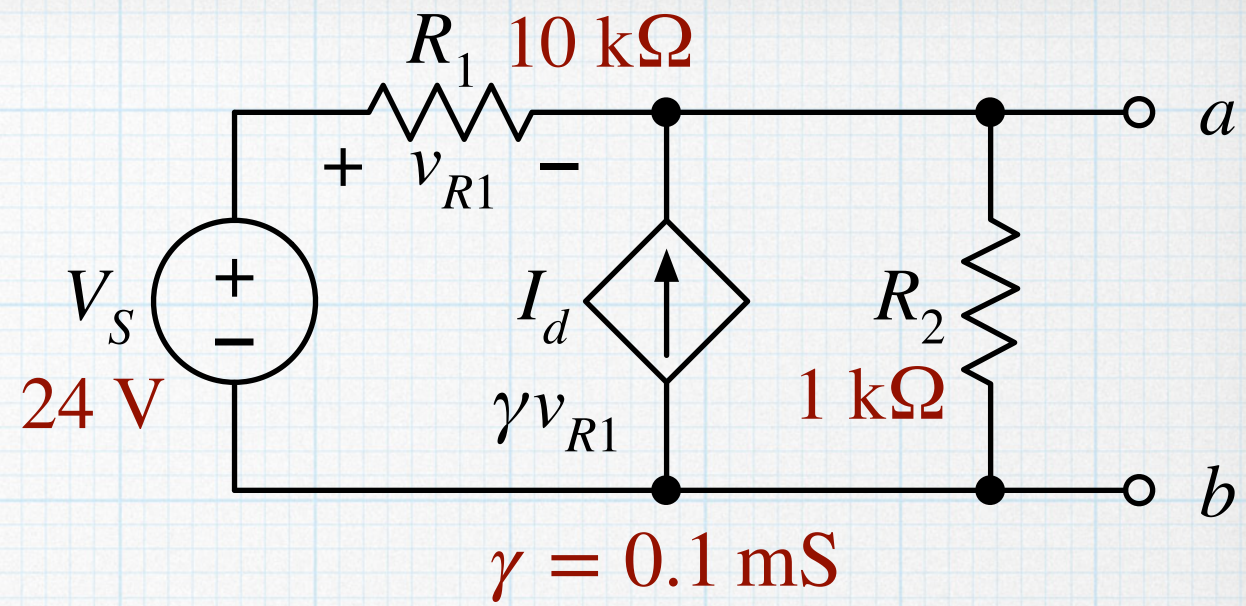
$$= \left(10\,\mathrm{k}\Omega\right) \| \left(10\,\mathrm{k}\Omega\right) \| \left(2.03\,\mathrm{k}\Omega\right)$$

$$= 1.44\,\mathrm{k}\Omega$$

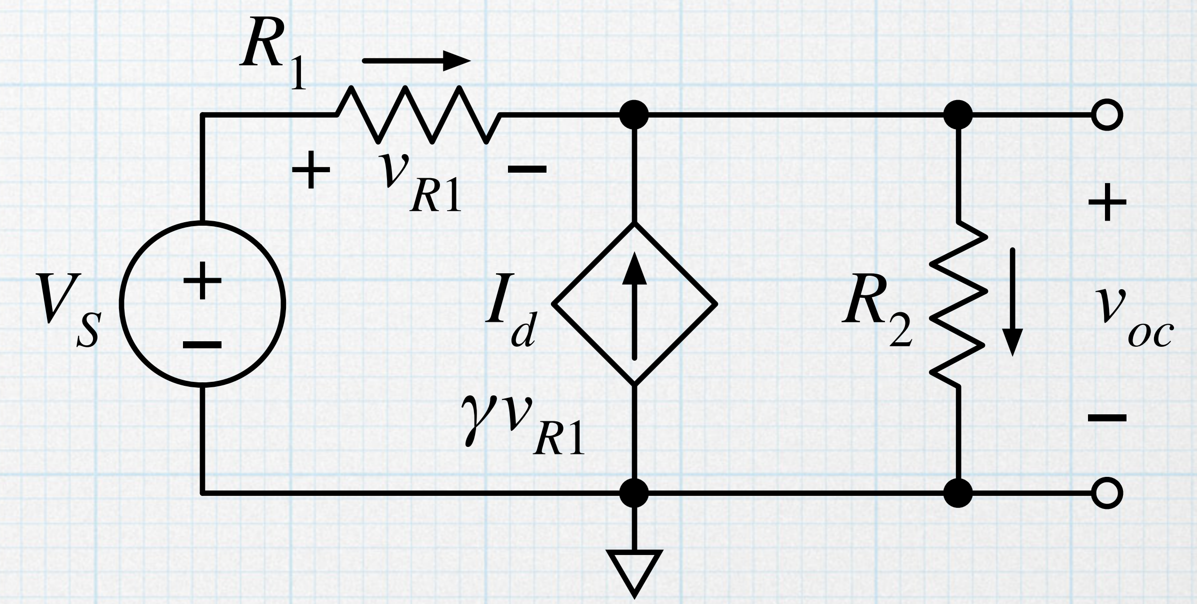
Confirming the previous calculation.

Example 7

Find the values for the Thevenin and Norton equivalents for the circuit at right.



Because this circuit has a dependent source, we will not be able to use the short-cut method to find equivalent resistance. We must find v_{oc} and i_{sc}.



Use NV to find v_{oc}.

$$\frac{V_s - v_{oc}}{R_1} + I_d = \frac{v_{oc}}{R_2}.$$

$$I_d = \gamma \cdot v_{R1} = \gamma \left(V_S - v_{oc} \right)$$

$$V_S - v_{oc} + \gamma R_1 \left(V_S - v_{oc} \right) = \frac{R_1}{R_2} v_{oc}$$

$$v_{oc} = \frac{V_S \left(1 + \gamma R_1 \right)}{1 + \gamma R_1 + \frac{R_1}{R_2}}$$

Inserting values

$$v_{oc} = \frac{(24\,\text{V}) \left[1 + (0.1\,\text{mS}) \left(10\,\text{k}\Omega \right) \right]}{\left[1 + (0.1\,\text{mS}) \left(10\,\text{k}\Omega \right) + \frac{10\,\text{k}\Omega}{1\,\text{k}\Omega} \right]}$$

$$= 4\,\text{V} = V_{Th}.$$

Example 7 (cont.)

Next, find i_{sc}. When shorting the output, R_2 is shorted out. We see that

$$i_{sc} = i_{R1} + \gamma v_{R1}$$

Since the upper right node is now grounded, $v_{R1} = V_S$:

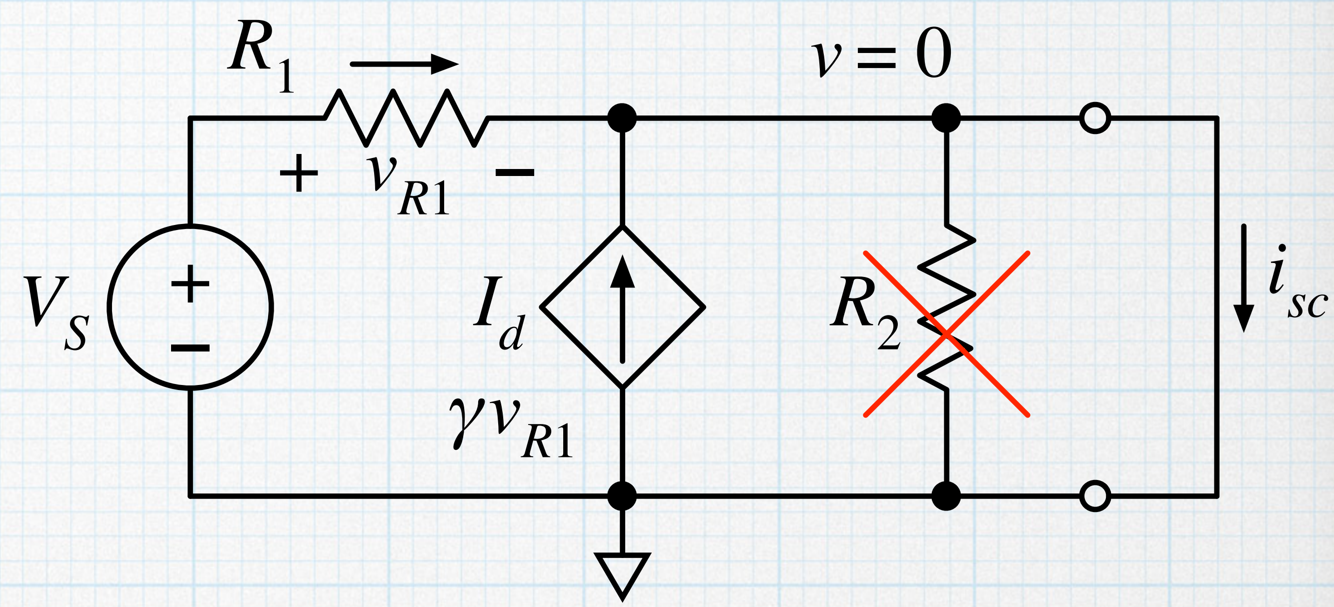
$$i_{sc} = \frac{V_S}{R_1} + \gamma V_S$$

$$= \frac{24\ \text{V}}{10\ \text{k}\Omega} + (0.1\ \text{mS})(24\ \text{V})$$

$$= 4.8\ \text{mA}$$

Finally, $R_{Th} = \dfrac{v_{oc}}{i_{sc}} = 833\ \Omega.$

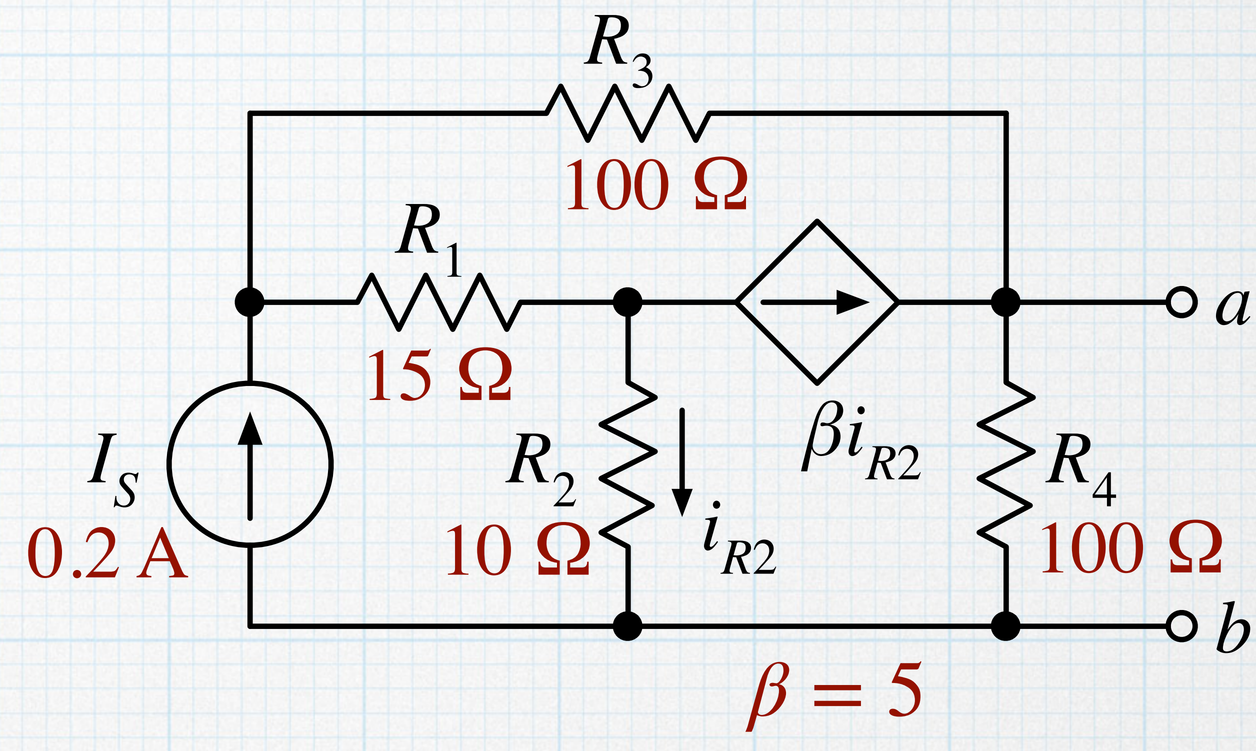
$V_{Th} = 4\ \text{V}, I_N = 4.8\ \text{mA}, R_{Th} = 833\ \Omega$



Note that if we had incorrectly tried to use the short-cut method for finding R_{Th}, we would have shorted the voltage source and then probably removed the dependent current source, leaving the two resistors in parallel. The result would have been "close-ish" to the actual value, but definitely not correct.

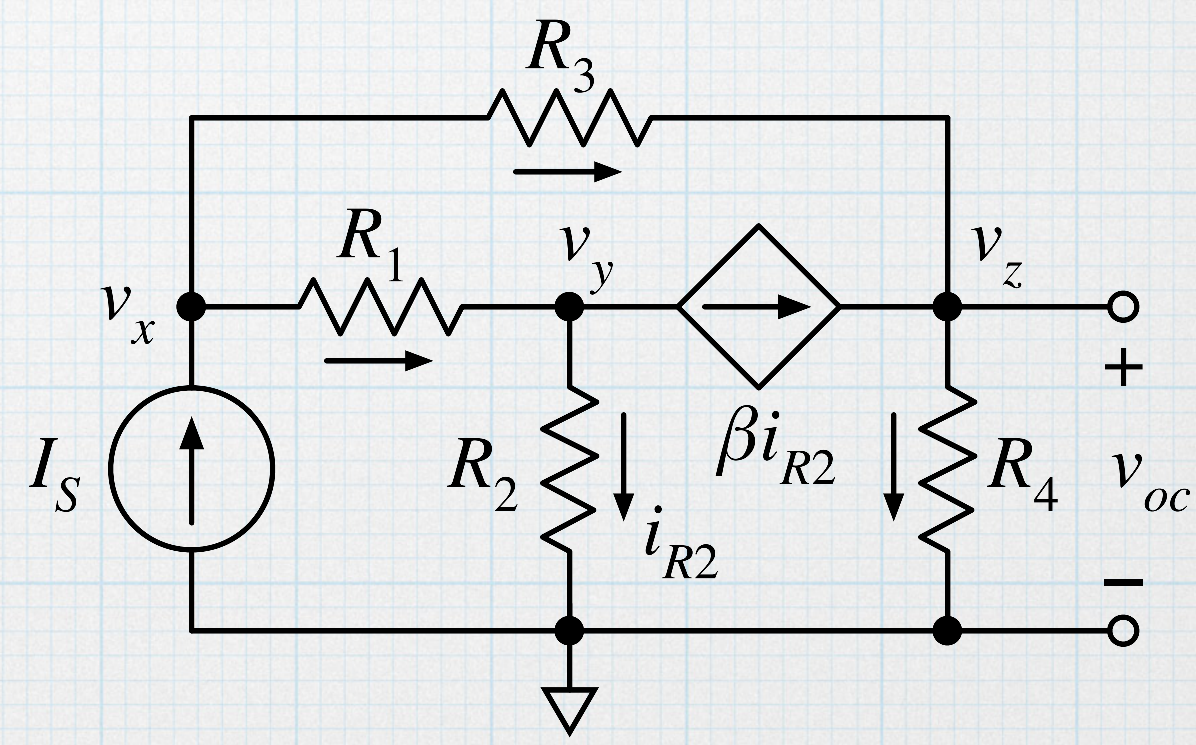
Example 8

Find the values for the Thevenin and Norton equivalents for the circuit at right.



Again, with the dependent source, we will not be able to use the short-cut method to find equivalent resistance. We must find v_{oc} and i_{sc}.

First v_{oc}. Hmmm. This is not a simple circuit. It needs three node voltages or two mesh currents with a supermesh. We will try NV.



$$I_S = i_{R1} + i_{R3}$$

$$i_{R1} = i_{R2} + I_d = i_{R2} + \beta i_{R2}$$

$$i_{R3} + \beta i_{R2} = i_{R4}$$

Example 8 (cont.)

The NV equations are:

$$I_S = \frac{v_x - v_y}{R_1} + \frac{v_x - v_z}{R_3}$$

$$\frac{v_x - v_y}{R_1} = \frac{v_y}{R_2} + \beta\frac{v_y}{R_2} = (\beta + 1)\,\frac{v_y}{R_2}$$

$$\frac{v_x - v_z}{R_3} + \beta\frac{v_y}{R_2} = \frac{v_z}{R_4}$$

Rearranging a bit:

$$\left(1 + \frac{R_1}{R_3}\right)v_x - v_y - \left(\frac{R_1}{R_3}\right)v_z = R_1 I_S$$

$$-v_x + \left[1 + \frac{R_1}{R_2}(1+\beta)\right]v_y = 0$$

$$-v_x - \beta\left(\frac{R_3}{R_2}\right)v_y + \left(1 + \frac{R_3}{R_4}\right)v_z = 0$$

Inserting values

$$1.15v_x - v_y - 0.15v_z = 3\text{ V}$$

$$-v_x + 10v_y = 0$$

$$-v_x - 50v_y + 2v_z = 0$$

We can use a solver directly. Or use the middle equation to eliminate v_x from the first and third equations.

$$10.5v_y - 0.15v_z = 3\text{ V}$$

$$-60v_y + 2v_z = 0$$

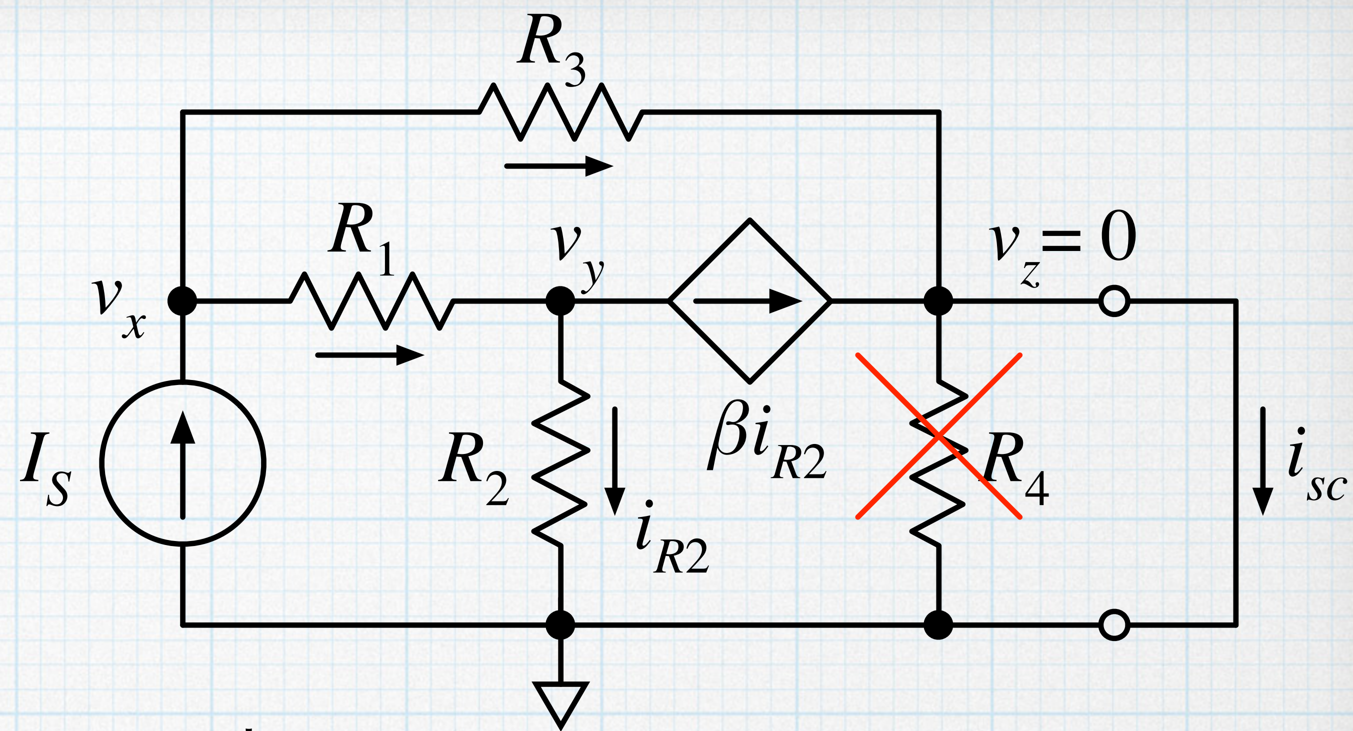
Solving the resulting 2x2 gives

$v_y = 0.5$ V and $v_z = 15$ V. And then $v_x = 5$ V.

Since $v_{oc} = v_z$,

$V_{Th} = v_z = 15$ V.

Example 8 (cont.)



Next, find i_{sc}. Shorting the output shorts R_4 as well. Then

$$i_{sc} = i_{R3} + \beta i_{R2}.$$

We need to find the new values for v_x and v_y. Use NV again.

$$I_S = \frac{v_x - v_y}{R_1} + \frac{v_x}{R_3}$$

$$\frac{v_x - v_y}{R_1} = (\beta + 1)\frac{v_y}{R_2}$$

$$\left(1 + \frac{R_1}{R_3}\right)v_x - v_y = R_1 I_S$$

$$-v_x + \left[1 + \frac{R_1}{R_2}(1+\beta)\right]v_y = 0$$

$$1.15 v_x - v_y = 3\,\text{V}$$

$$-v_x + 10 v_y = 0$$

Solving gives $v_x = 2.86$ V and $v_y = 0.286$ V.

Then

$$i_{R3} = \frac{2.86\,\text{V}}{100\,\Omega} = 28.6\,\text{mA}$$

and $i_{R2} = \dfrac{0.286\,\text{V}}{10\,\Omega} = 28.6\,\text{mA}.$

Finally,

$i_{sc} = 28.6$ mA + (5)(28.6 mA) = 0.172 A.

At last! We have arrived:

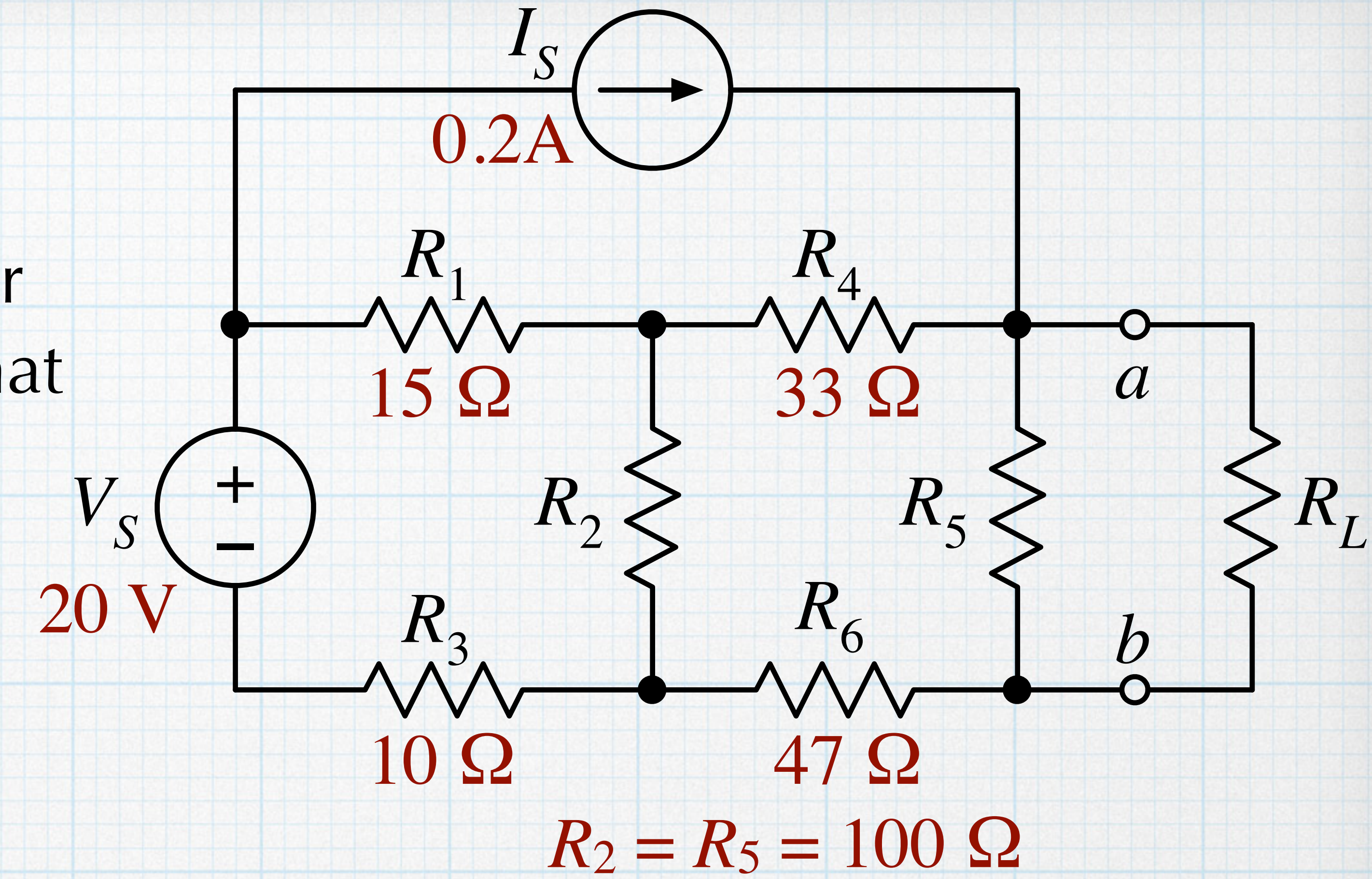
$V_{Th} = 15$ V, $I_N = 0.172$ A, and

$$R_{Th} = \frac{15\,\text{V}}{0.172\,\text{A}} = 87\,\Omega$$

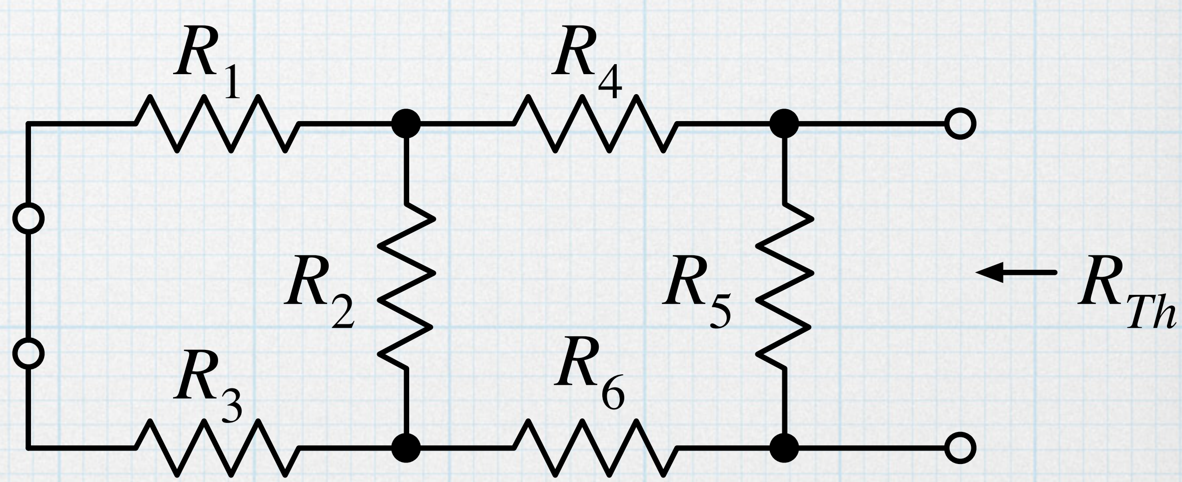
Example 9

Find the value of load resistance that will result in maximum power delivered to the load. Calculate that maximum power.

(Obviously, this is another way of saying "Find the Thevenin or Norton equivalent of the circuit.")



Since a key to finding the max power is finding the resistance, and since this circuit has only independent sources, we could begin by using the short-cut method to find R_{Th}.



After deactivating the sources, the equivalent resistance seen from the output port is straight forward.

$$R_{Th} = R_5 \Big\| \left[R_4 + R_6 + R_2 \Big\| \left(R_1 + R_3 \right) \right]$$

$$= (100\,\Omega) \Big\| \left[80\,\Omega + (100\,\Omega) \Big\| (25\,\Omega) \right]$$

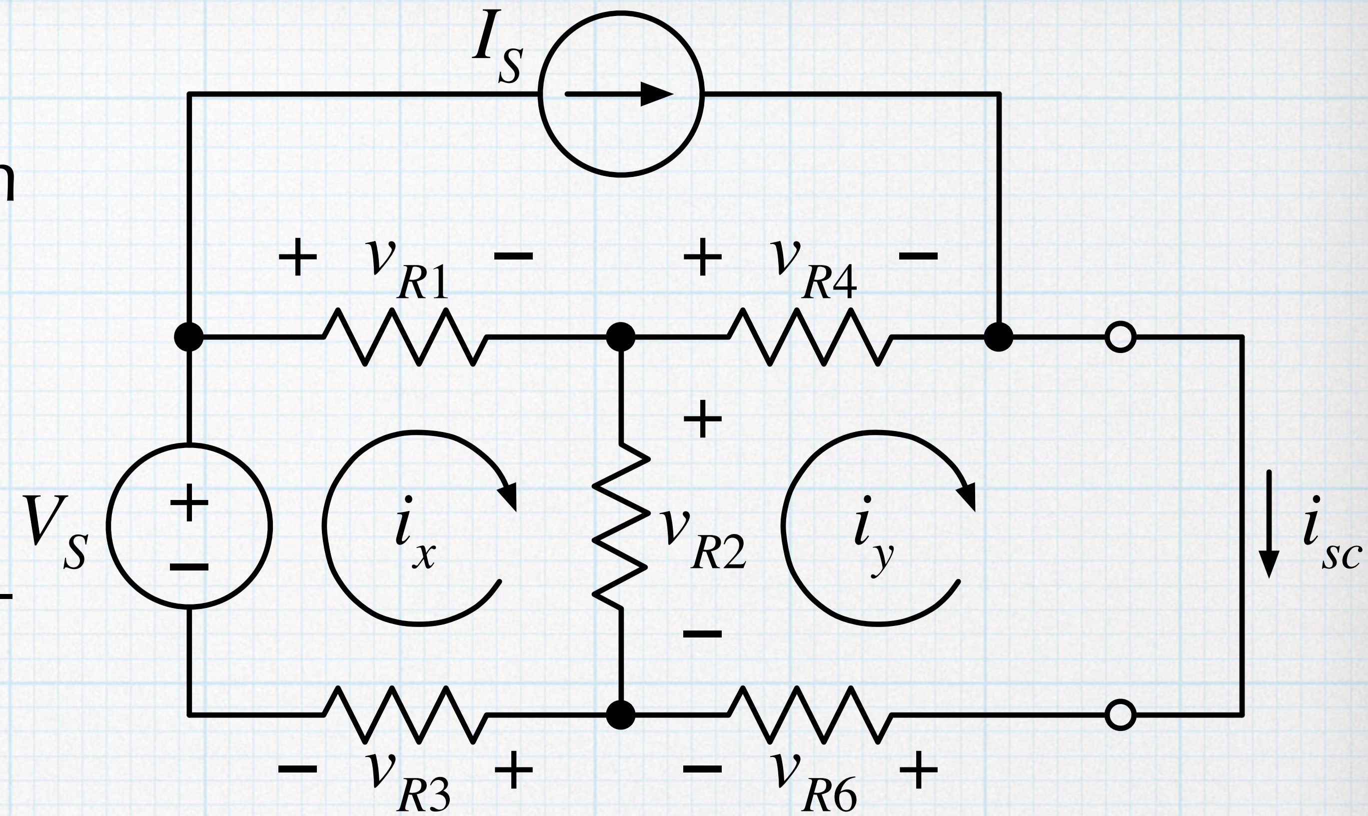
$$= 50\,\Omega$$

So the load resistance should be 50 Ω for maximum power.

Example 9 (cont.)

However, to calculate the maximum power, we also need the Thevenin voltage (or Norton current.)

As always, we can choose the analysis method. Let's use the mesh-current method to find the Norton current. With the port shorted, R_5 is shorted. With the circuit set up for MC analysis, we see that $i_{sc} = i_y$.



$$V_S - v_{R1} - v_{R2} - v_{R3} = 0$$

$$v_{R2} - v_{R4} - v_{R6} = 0$$

$$V_S - R_1\left(i_x - I_S\right) - R_2\left(i_x - i_y\right) - R_3 i_x = 0$$

$$R_2\left(i_x - i_y\right) - R_4\left(i_y - I_S\right) - R_6 i_y = 0$$

$$\left(R_1 + R_2 + R_3\right) i_x - R_3 i_y = V_S + R_1 I_S$$

$$-R_2 i_x - \left(R_2 + R_4 + R_6\right) i_y = R_4 I_S$$

Insert values:

$$\left(125\,\Omega\right) i_x - \left(100\,\Omega\right) i_y = 23\text{ V}$$

$$-\left(100\,\Omega\right) i_x + \left(180\,\Omega\right) i_y = 6.6\text{ V}$$

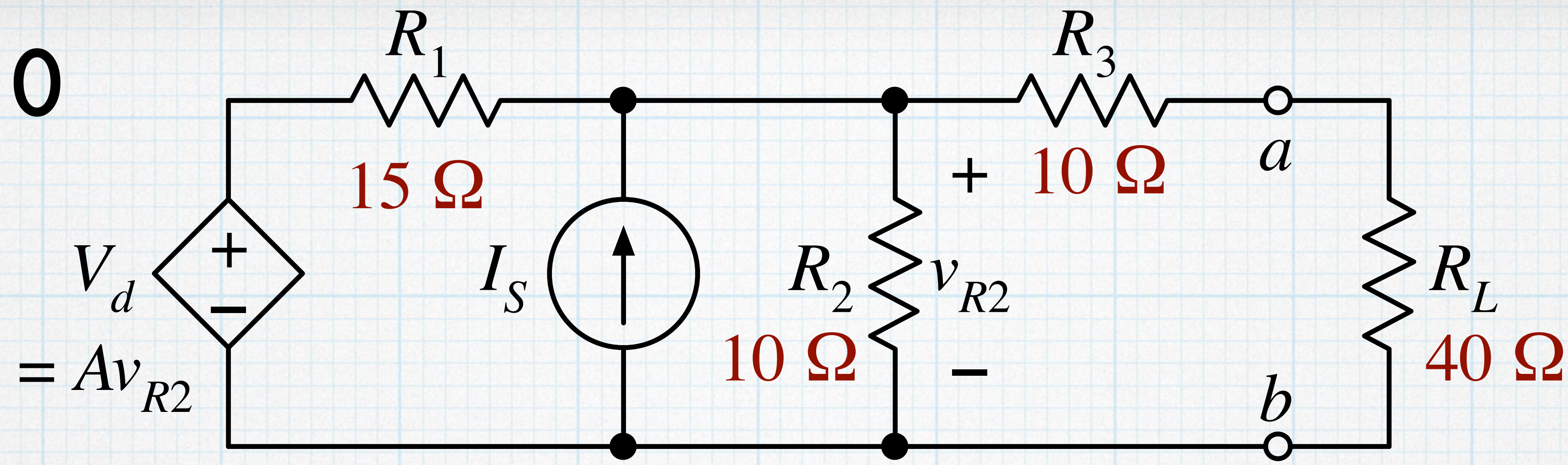
Solving:

$i_x = 0.384$ A and $i_y = 0.25$ A

$$i_{sc} = 0.25\text{ mA} \rightarrow V_{Th} = 12.5\text{ V}$$

$$P_{max} = \frac{\left(6.25\text{ V}\right)^2}{50\,\Omega} = 0.781\text{ W}$$

Example 10



We are tasked with finishing the design of a small circuit that will be part of some electronic system. The basic topology is shown above. The circuit must have its equivalent resistance matched to the external 40-Ω load and be able to deliver 3.6 W to the load. Some of the internal resistors are already specified. Finish the design by choosing I_S for the current source and A for the dependent voltage source so that the circuit is delivering its maximum available power to the load.

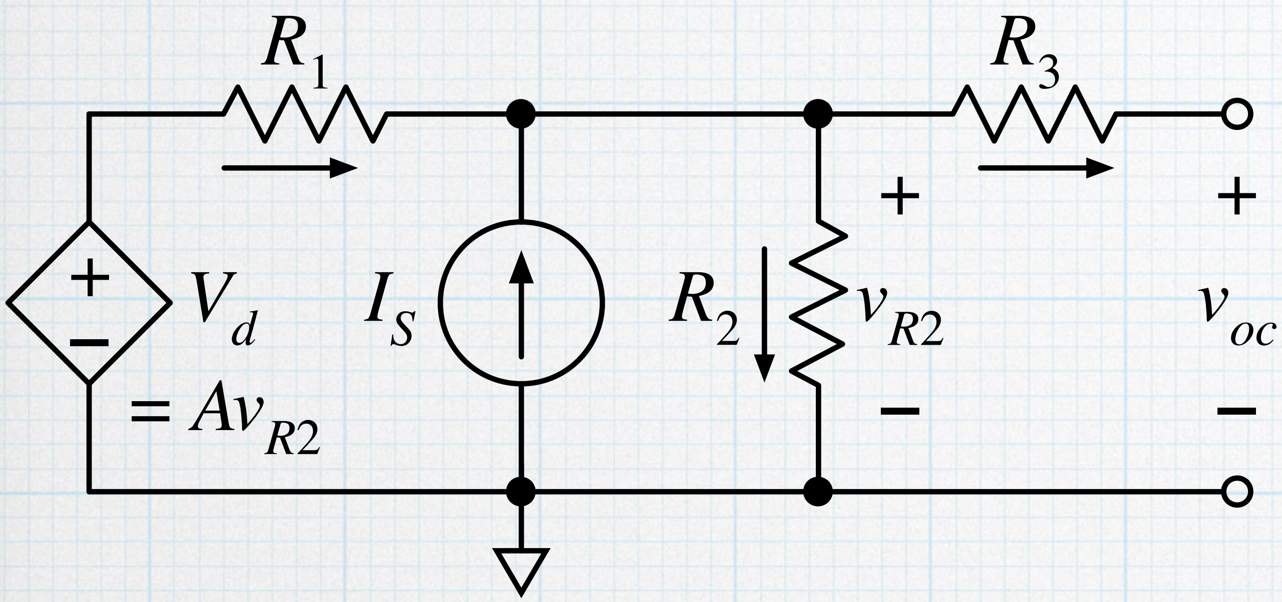
As stated, R_{Th} for the circuit should be equal to 40 Ω. Also,

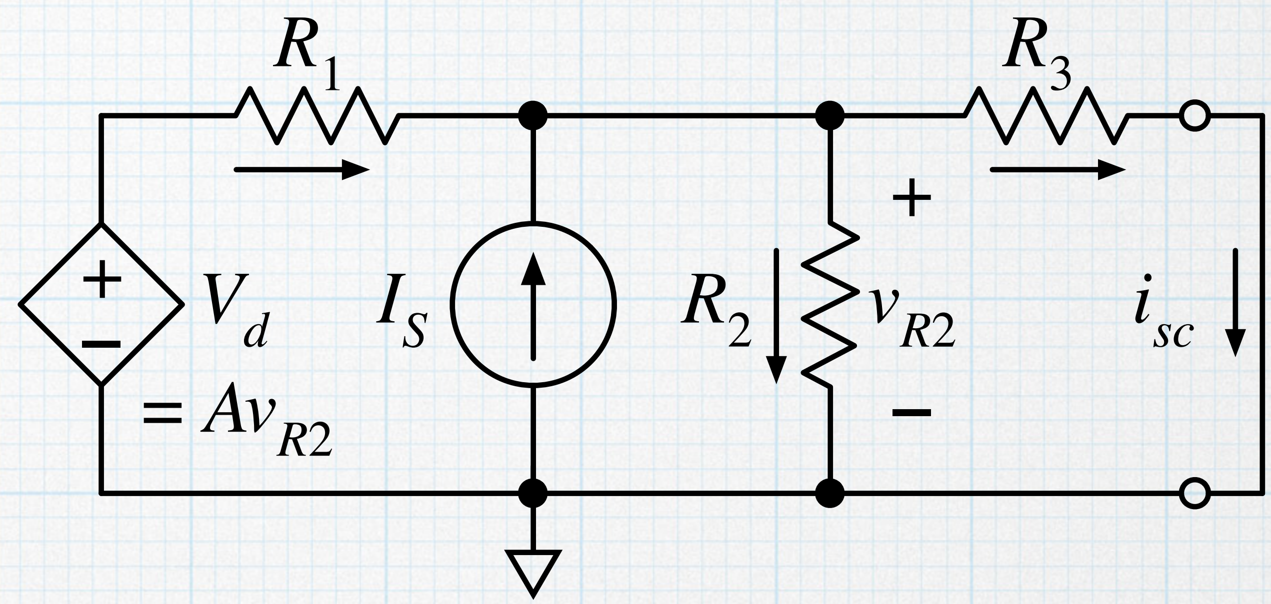
$$v_{RL} = \sqrt{P \cdot R_L} = \sqrt{(3.6\,\text{W})\,(40\,\Omega)} = 12\,\text{V}.$$

Since at max power, $v_{RL} = V_{Th}/2$, then $V_{Th} = 24$ V.

So we can find expressions for v_{oc} and i_{sc}, and then choose values so that the Thevenin equivalent of the circuit meets the requirements.

Example 10 (cont.)





Find v_{oc}, which is equal to v_{R2}.

$$\frac{Av_{oc} - v_{oc}}{R_1} + I_S = \frac{v_{oc}}{R_2}$$

$$v_{oc} = \frac{R_1 I_S}{1 + \frac{R_1}{R_2} - A}$$

Find i_{sc}, which is equal to i_{R3}.

$$\frac{Av_{R2} - v_{R2}}{R_1} + I_S = \frac{v_{R2}}{R_2} + \frac{v_{R2}}{R_3}$$

$$v_{R2} = \frac{R_1 I_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} - A}$$

$$i_{sc} = i_{R3} = \frac{v_{R2}}{R_3} = \frac{\frac{R_1}{R_3} I_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} - A}$$

Example 10 (cont.)

$$v_{oc} = \frac{R_1 I_S}{1 + \frac{R_1}{R_2} - A} \quad \text{and} \quad i_{sc} = \frac{\frac{R_1}{R_3} I_S}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} - A}$$

We know that i_{sc} must be equal to v_{oc} / R_{Th} = (24 V)/(40 Ω) = 0.6 A, and so we could treat the pair above as two equations in two unknowns (I_S and A) and solve in the usual fashion. But the form of the two expressions suggests a possibly easier path. Dividing the left by the right:

$$\frac{v_{oc}}{i_{sc}} = R_{Th} = \frac{R_3 \left(1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} - A\right)}{1 + \frac{R_1}{R_2} - A}$$

I_S has disappeared!

$$40\,\Omega = \frac{(10\,\Omega)(4 - A)}{2.5 - A}$$

A bit of algebraic finagling gives

$A = 2$.

Now, knowing A, we can use the v_{oc} expression to find I_S:

$$I_S = \frac{v_{oc}\left(1 + \frac{R_1}{R_2} - A\right)}{R_1}$$

$$= \frac{(24\text{ V})\left(0.5\right)}{15\,\Omega} = 0.8\text{ A}$$

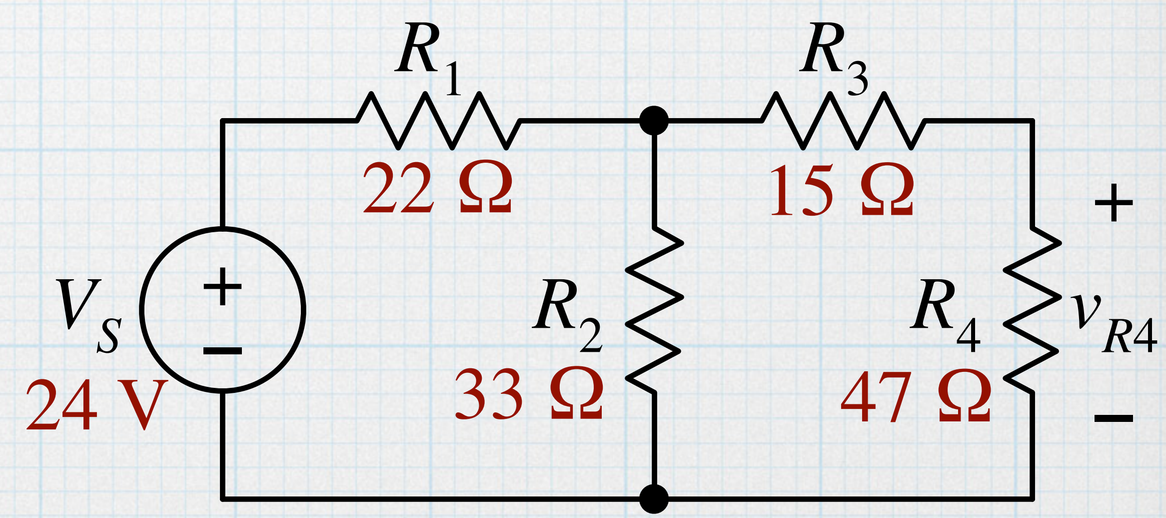
Design complete!

Example 11 — Thevenin as an analysis tool

In the right circumstance, it is possible to use a Thevenin/Norton equivalent as circuit-analysis tool. Usually, this involves recognizing a piece of a circuit for which we already know the equivalent, and then substituting in to simplify subsequent calculations.

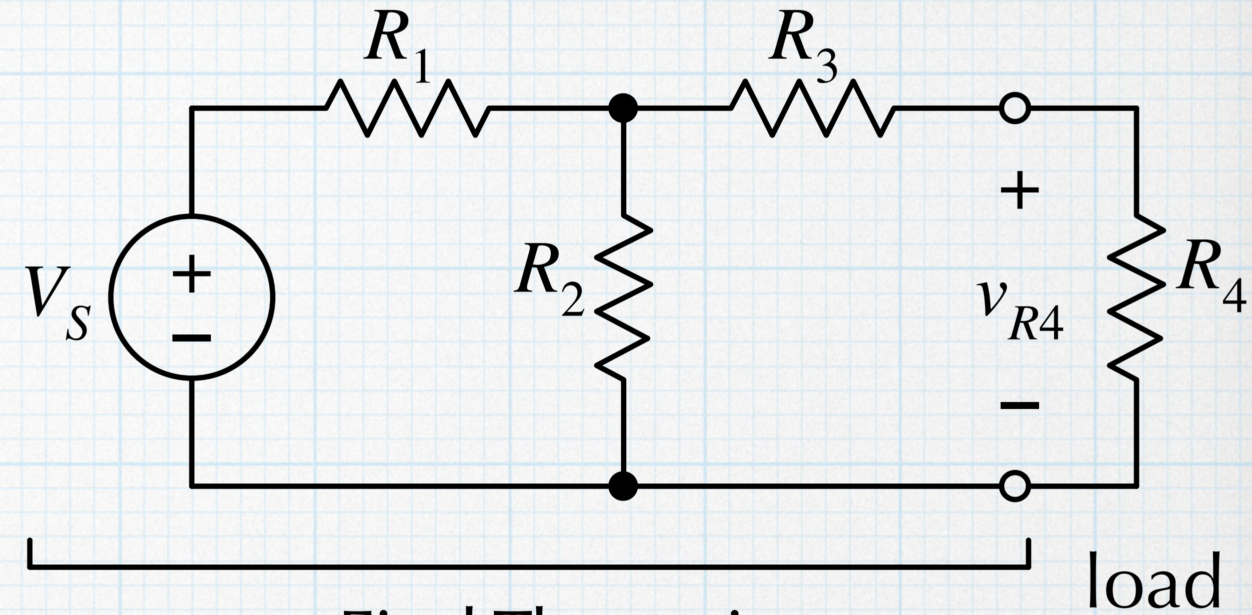
Note how this contrasts with previous Thevenin/Nortion examples, where we used techniques from our circuit-analysis toolbag to find the voltages and currents needed to determine an equivalent. Now we circle around and use an equivalent to find a voltage or current.

As a simple example, consider the ladder circuit shown, where we would like to find v_{R4}. Earlier, we analyzed this circuit using several different methods — cascaded voltage dividers, node voltages, and mesh currents. Using a Thevenin equivalent gives us another option.



Example 11 (cont.)

Taking a slightly different view of the circuit, we might treat R_4 as a load attached to a port. Then we can find the Thevenin of the rest of the circuit with respect to that port.

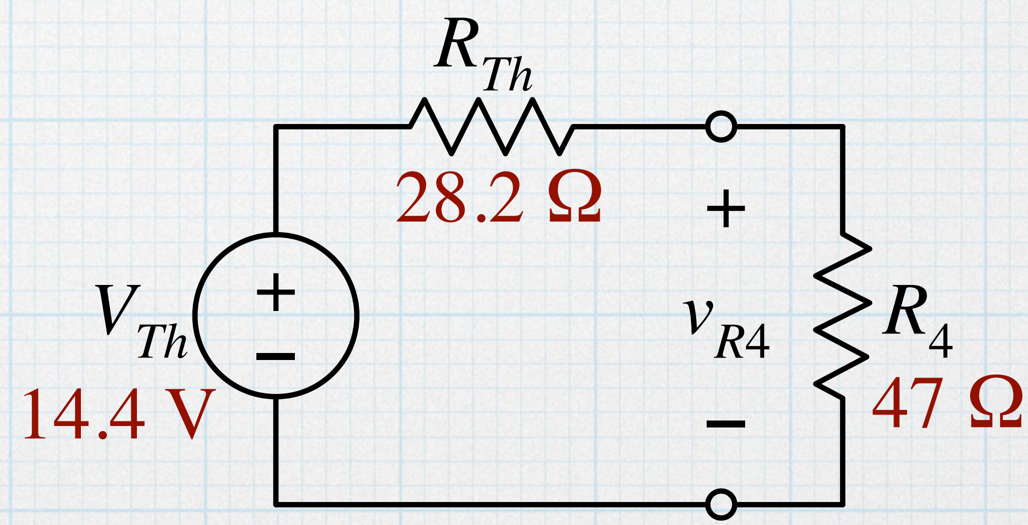


Find Thevenin.

Recall that we found the Thevenin of this particular circuit back in Example 3 — we already know the result:

$$V_{Th} = \frac{R_2}{R_1 + R_2} V_S = 14.4\ \text{V and}$$

$$R_{Th} = R_3 + R_1 \| R_2 = 28.2\ \Omega$$



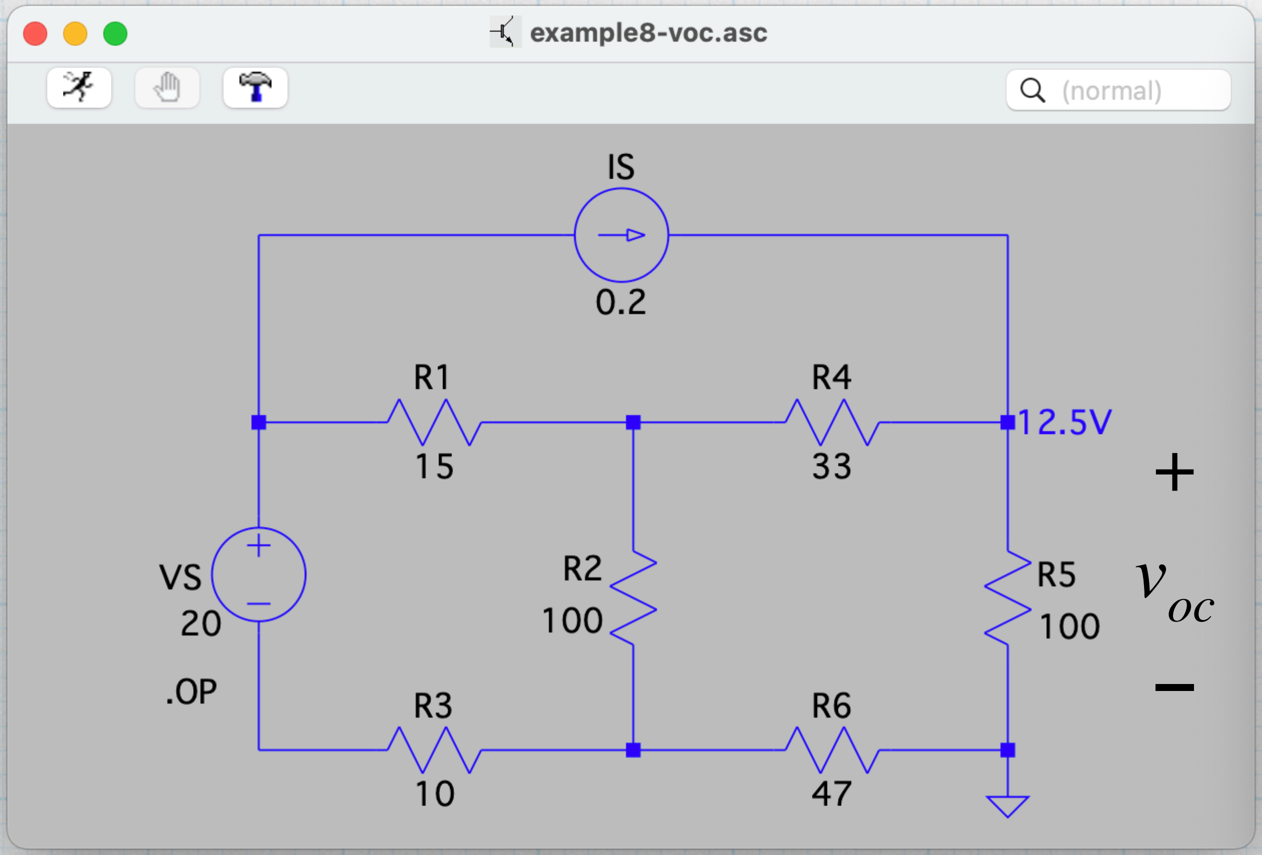
Then

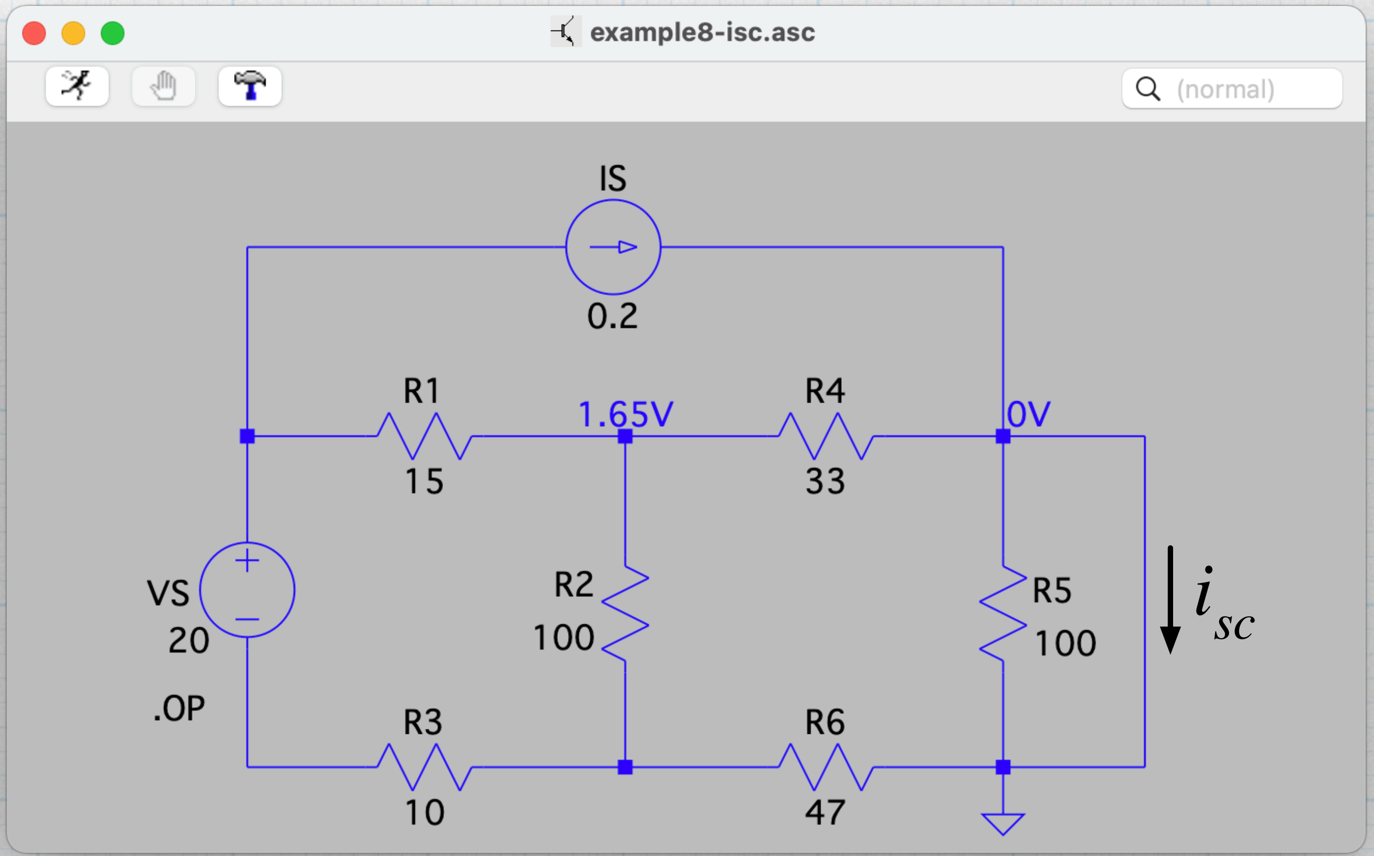
$$v_{R4} = \frac{R_4}{R_{Th} + R_4} V_{Th} = 9\ \text{V}$$

Q.E.D.

Example 12 (Ex. 9 redux)

Of course, we can also use SPICE to find the equivalent circuits. Below are simulations using LTspice for the circuit of Example 8. (Note the placement of the ground, making it easy to display the desired voltages.)





$V_{Th} = v_{oc} = 12.5$ V.

$I_N = i_{sc} = 0.2\text{ A} + (1.65\text{ V})/(33\ \Omega) = 0.25$ A.

(Note: These simulations were done with the Mac version of LTspice, which is somewhat hamstrung in terms of displaying DC currents. If using the Windows version, a better way to show i_{sc} would be to attach a voltage source with value 0 V (same as a short circuit) across R_5 and then display the current through the "short source".)