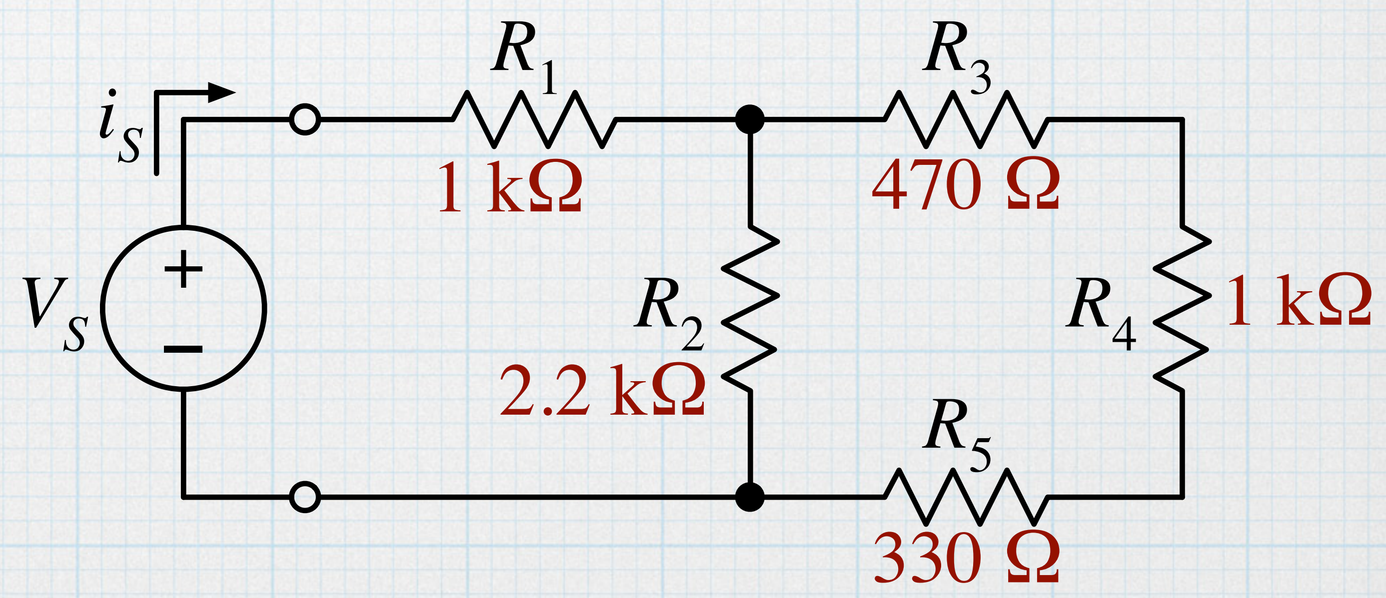
Series and parallel combinations

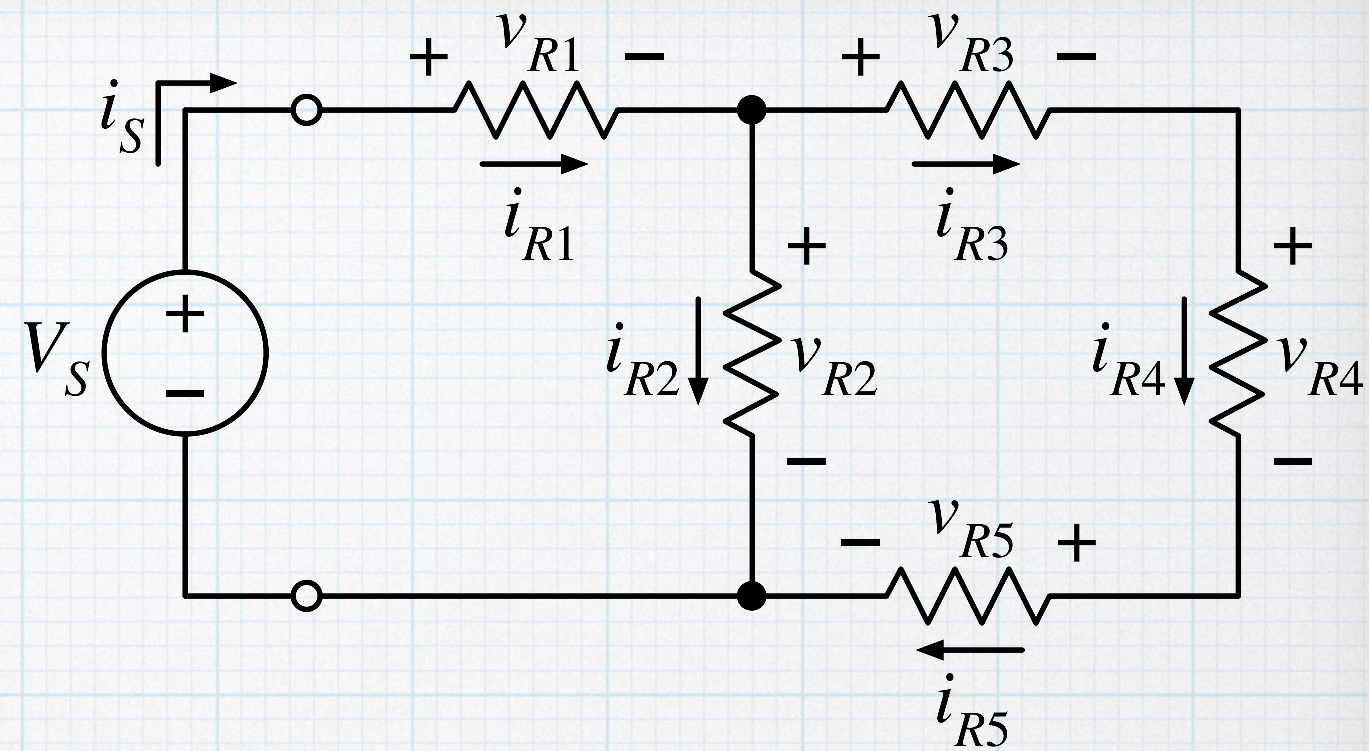
One of the simplest and most useful things we can do in a circuit is to reduce the complexity by combining similar elements that have series or parallel connections. Resistors, voltage sources, and current sources can all be combined and replaced with equivalents in the right circumstances.

We start with resistors. In many situations, we can reduce complex resistor networks down to a few, or even a single, equivalent resistance. As always, the exact approach depends on what we want to know about the circuit, but resistor reduction is a tool that we will use over and over.

To set the stage, consider the circuit at right. We might like to determine the power from the source, which requires knowing the current. Of course, we don't know the source current initially — we must find it by finding the current flowing in the resistors.



In the circuit, $i_S = i_{R1}$, so our goal is to find that. Set to work with Kirchoff's Laws. Since we don't know anything at the outset, we will have to come up with enough equations to have a simultaneous set that can be solved.



KCL: $i_{R1} = i_{R2} + i_{R3}$; $i_{R3} = i_{R4} = i_{R5}$.

KVL: $V_S - v_{R1} - v_{R2} = 0$; $v_{R2} - v_{R3} - v_{R4} - v_{R5} = 0$.

Using Ohm's Law to write voltages in terms of currents and then fiddling around to reduce the equations to a manageable set, we arrive at three equations relating, i_{R1}, i_{R2}, and i_{R3}. (We are skipping all the details here — there will be plenty of time for developing simultaneous equations later.)

$$i_{R1} = i_{R2} + i_{R3}$$

$$V_S - i_{R1}R_1 - i_{R2}R_2 = 0$$

$$i_{R2}R_2 - i_{R1}(R_3 + R_4 + R_5) = 0.$$

Three equations, three unknowns.

$$i_{R1} = i_{R2} + i_{R3}$$

$$V_S - i_{R1}R_1 - i_{R2}R_2 = 0$$

$$i_{R2}R_2 - i_{R1}(R_3 + R_4 + R_5) = 0.$$

Soon enough, we will be adept at handling problems like this. For now, we will put our trust in Wolfram-Alpha (or something similar), and let it grind out the answers.

$$i_{R1} = 5.02 \text{ mA}.$$

$$i_{R2} = 2.26 \text{ mA}.$$

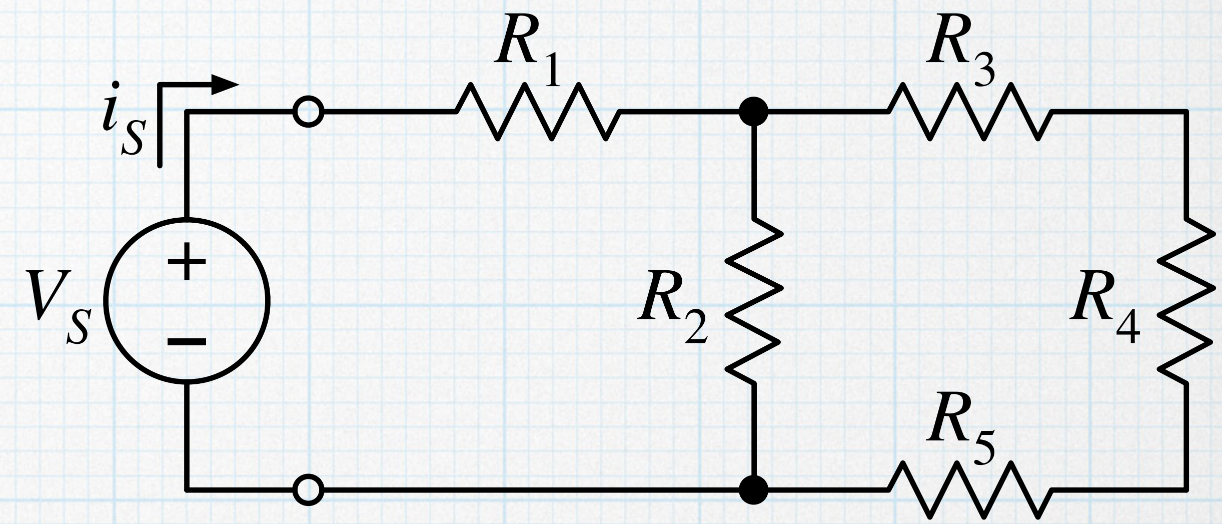
$$i_{R3} = 2.76 \text{ mA}.$$

Finally, $i_{R1} = i_S$ and the power being delivered by the source is $P_S = V_S \cdot i_S = (10 \text{ V})(5.02 \text{ mA}) = 50.2 \text{ mW}$.

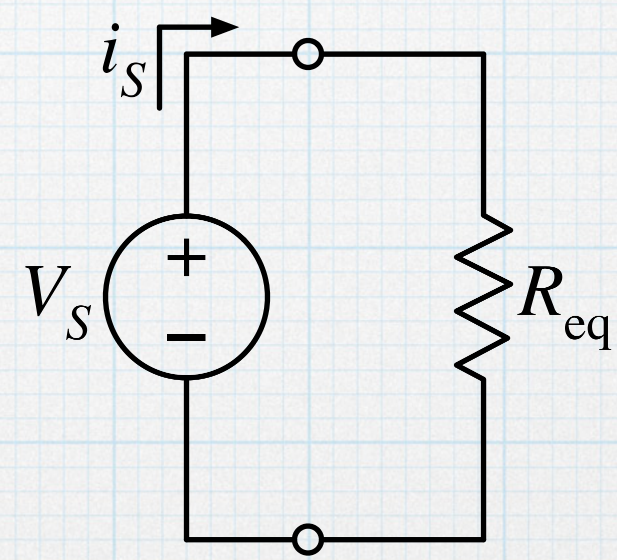
However, this business of finding three equations in three unknowns and solving all that seems a lot of work to determine one number in a relatively simple circuit. Is there a simpler way? Of course, the answer is "yes".

Equivalent Resistance

The original circuit was a single source with a network of resistors attached. The resistor currents are related to the source current by KCL. The resistor voltages are related to the source voltage by KVL. The resistor currents are related to the resistor voltages by Ohm's Law.



Then it seems reasonable that the source voltage and source current should be related by Ohm's Law, meaning that there must be some *equivalent resistance* that represents the cumulative effect of resistors in the network:

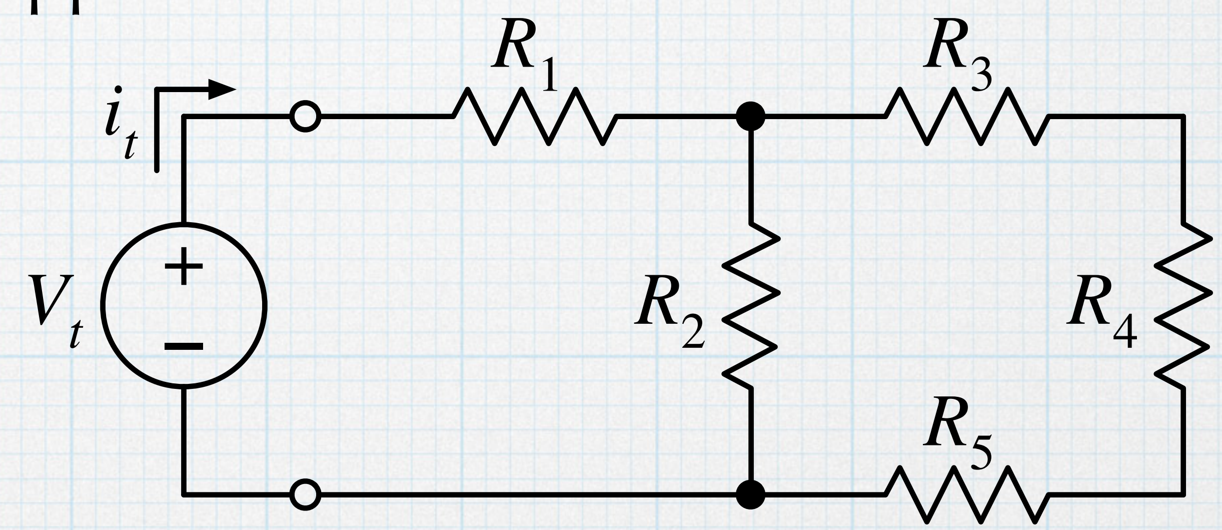


$$R_{eq} = \frac{V_S}{i_S}$$

Equivalent Resistance

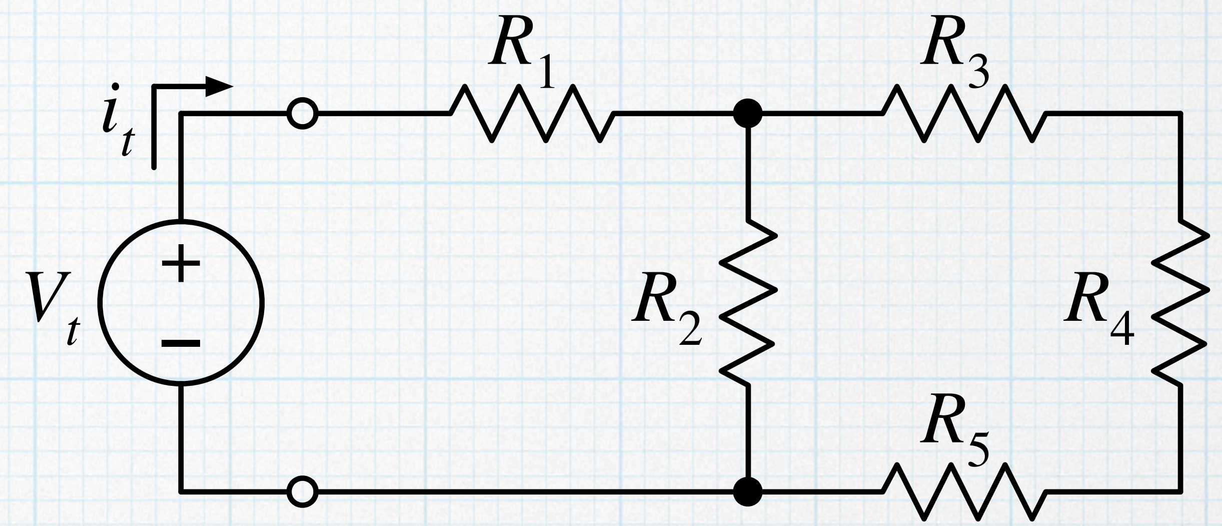
The question is how to find the equivalent resistance of the network. The general approach would be to apply a "test generator" to the network. A test generator is a voltage or current source with a value that we can choose. For example, if we apply a test voltage source with value V_t, as shown below, then we can calculate the current, i_t, that flows into the network due to the applied source.

The equivalent resistance would then be $R_{eq} = \frac{V_t}{i_t}$.



In lab we could something similar by building the circuit, applying a test voltage, and measuring the result current. In lab, this process goes by a different name — it's called "using an ohmmeter".

Of course, we have already done this. The earlier calculation is identical to this test generator idea if we set V_t = 10 V. In the calculation, we found the current to be 5.02 mA. Then the equivalent resistance is



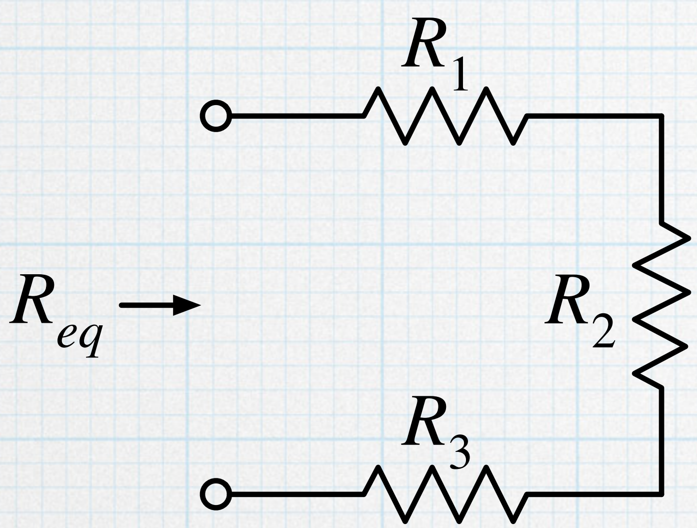
R_{eq} = 10 V / 5.02 mA = 1.99 kΩ.

However, this seems a bit pointless, because finding equivalent resistance using a test generator was as much work as finding the source current directly. In fact, it took one extra step to find the equivalent resistance.

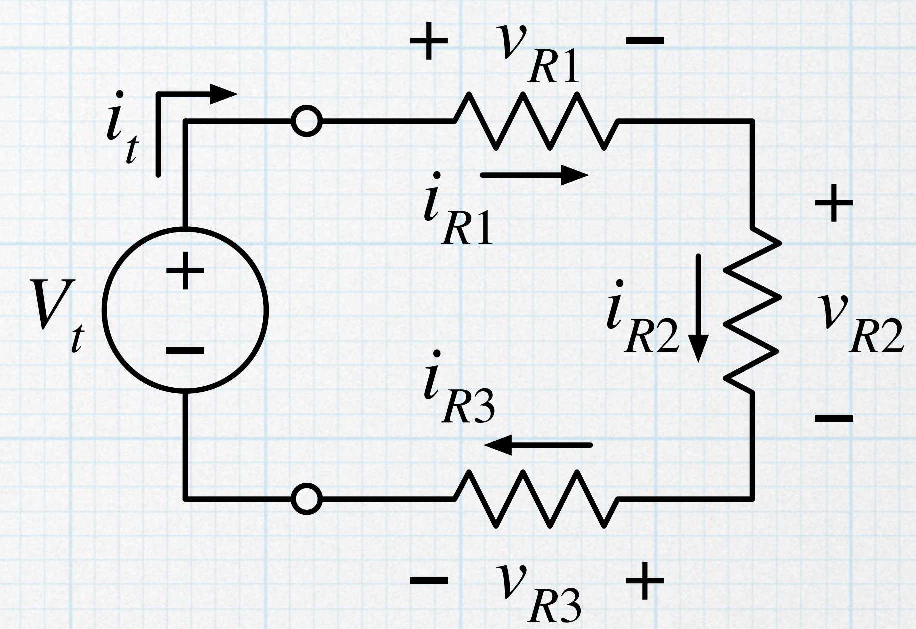
But fear not. We can start with simple relationships for the equivalent resistance of series and parallel combinations. Then we can use series and parallel combinations to break down complex resistor networks and analyze them in a piecemeal fashion. We will see that the equivalent resistance idea is simple to implement in most cases and can be a powerful method for analyzing circuits. We will use it repeatedly as move through EE 201 and 230.

Series combination

Resistors are in series, meaning that the same current flows in all.



Apply test voltage.
Define voltages and currents.



By KCL: $i_{R1} = i_{R2} = i_{R3} = i_t$ Expected, since they are in series.

By KVL: $V_t - v_{R1} - v_{R2} - v_{R3} = 0.$

Use Ohm's law to write voltages in terms of currents.

$$V_t - i_{R1}R_1 - i_{R2}R_2 - i_{R3}R_3 = 0$$

$$V_t - i_tR_1 - i_tR_2 - i_tR_3 = i_t\left(R_1 + R_2 + R_3\right) = 0$$

$$R_{eq} = \frac{V_t}{i_t} = R_1 + R_2 + R_3$$

Series combination

The equivalent resistance of resistors in series is simply the sum of the individual resistance.

$$R_{eq} = \sum_{m=1}^{N} R_m$$

The calculation is easy.

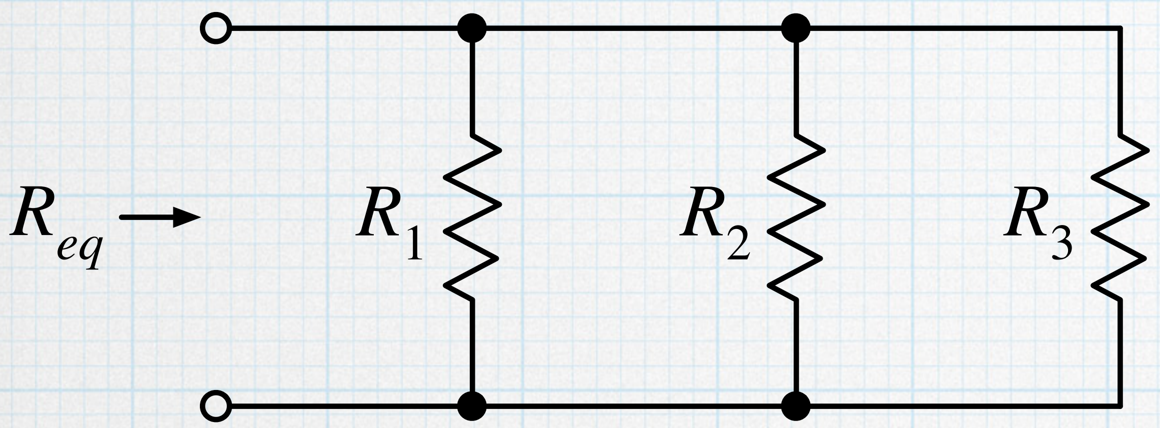
The equivalent resistance is always bigger than any of the individual resistors, $R_{eq} > R_m$.

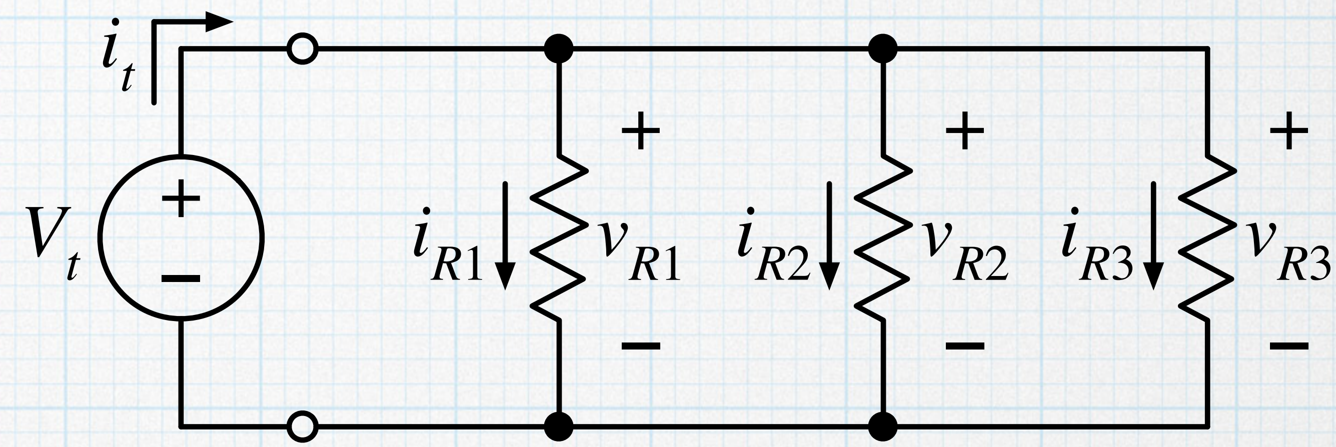
In fact, if one resistor is much much bigger than the rest, the equivalent resistance will be approximately equal to the one big resistor. For example, in the three-resistor string on the previous page, if R_1 = 10 kΩ, R_2 = 100 Ω, and R_3 = 1 Ω, then R_{eq} = 10.101 kΩ ≈ 10 kΩ.

This is why we can ignore the resistance of wires in most cases. Consider a 1-kΩ resistor with its two leads. If the resistor body has R_B = 1 kΩ and the wires are each R_w ≈ 0.01 Ω, the series equivalent resistance of the whole is resistor is then 1.00002 kΩ. In almost all practical cases, the wire resistance is negligible.

Parallel combination

Resistors in parallel — they all have the same voltage across.





Apply the test voltage.
Define voltages and currents.

By KVL: $v_t = v_{R1} = v_{R2} = v_{R3}$. Expected, since they are all in parallel

By KCL: $i_t = i_{R1} + \mathrm{i}_{\mathrm{R2}} + i_{R3}$. Use Ohm's law to write i_R in terms of v_R.

$$i_t = \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} + \frac{v_{R3}}{R_3}$$

$$i_t = \frac{v_t}{R_1} + \frac{v_t}{R_2} + \frac{v_t}{R_3} = v_t\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{1}{R_{eq}} = \frac{i_t}{v_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Parallel combination

The inverse of the equivalent resistance is equal to the sum of the inverses of all the resistance in the parallel combination.

$$\frac{1}{R_{eq}} = \sum_{m=1}^{N} \frac{1}{R_m}$$

The equivalent resistance will always be smaller than the resistance of any individual branch: $R_{eq} < R_m$ for all m.

If one resistor is much smaller than all other resistors in the parallel combination, (so that its inverse is much bigger), then the equivalent resistance will be approximately equal to that of the smallest resistor. For example, if the three parallel resistors from the previous page had values of $R_1 = 10\ \mathrm{k\Omega}$ ($1/R_1 = 10^{-4}\ \Omega^{-1}$), $R_2 = 100\ \Omega$ ($1/R_2 = 10^{-2}\ \Omega^{-1}$), and $R_3 = 1\ \Omega$ ($1/R_3 = 1\ \Omega^{-1}$), then $R_{eq} = 0.99\ \Omega \approx 1\ \Omega$ ($1/R_{eq} = 1.0101\ \Omega^{-1}$).

In fact, if we place a wire ($R_w \approx 0$) in parallel with other resistors, the equivalent resistance approximately zero — the wire has shorted out everything else.

Since the calculation for parallel resistors, with the need for inverses, can be a bit messy, there are some short-cuts that can used for special cases.

If there are only two resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2}{R_1 R_2} + \frac{R_1}{R_1 R_2} = \frac{R_1 + R_2}{R_1 R_2}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$ (Product over sum, which might be easier to compute.)

Two identical resistors, $R_1 = R_2 = R$:

$$R_{eq} = \frac{R^2}{R + R} = \frac{R}{2}$$ (e.g. Two 1-kΩ resistors in parallel gives 0.5 kΩ.)

N identical resistors in parallel (extending the idea):

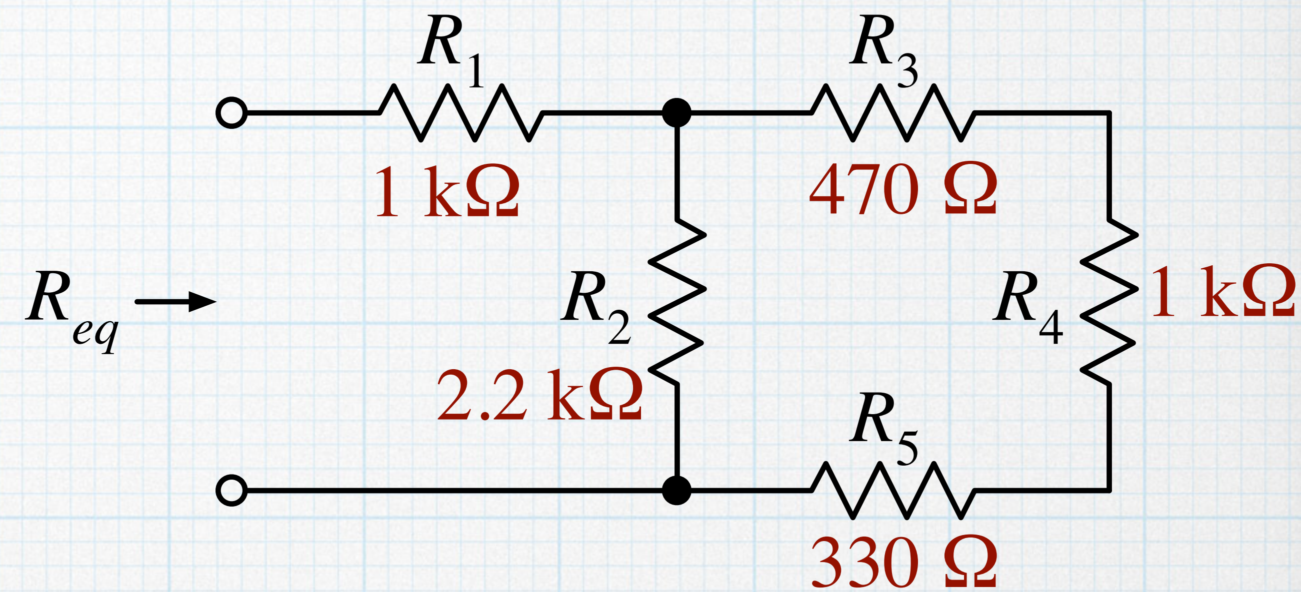
$$R_{eq} = \frac{1}{\frac{1}{R} + \frac{1}{R} + \ldots + \frac{1}{R}} = \frac{R}{N}.$$

If one resistor is much smaller than the rest ($R_1 << R_m$) (to re-emphasize)

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N}} \approx R_1$$ If $R_1 = 0$ (short circuit), then $R_{eq} = 0$.

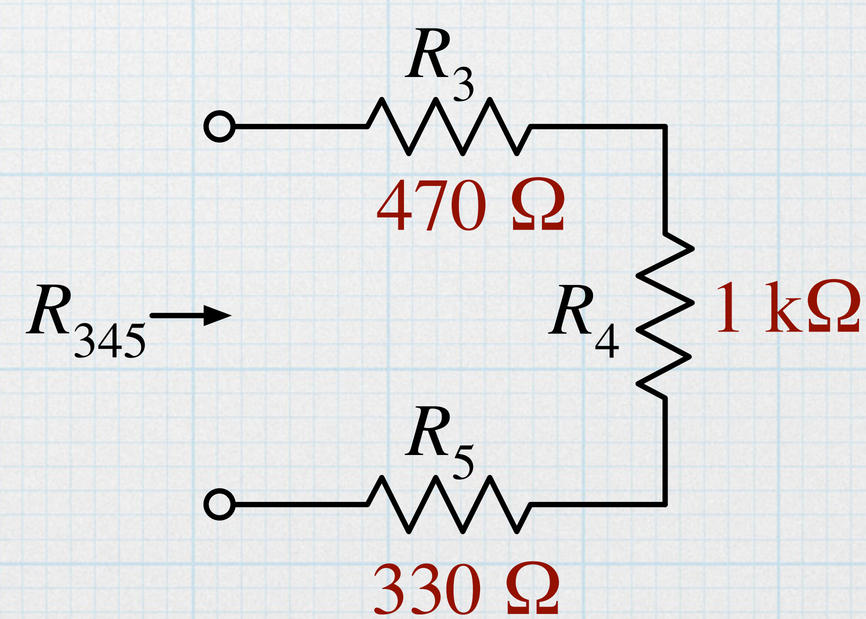
Breaking down networks using series and parallel

But not all circuits are simple combinations of series or parallel resistors. The initial example circuit clearly has some things that are in series and some elements that have a parallel-type connection.

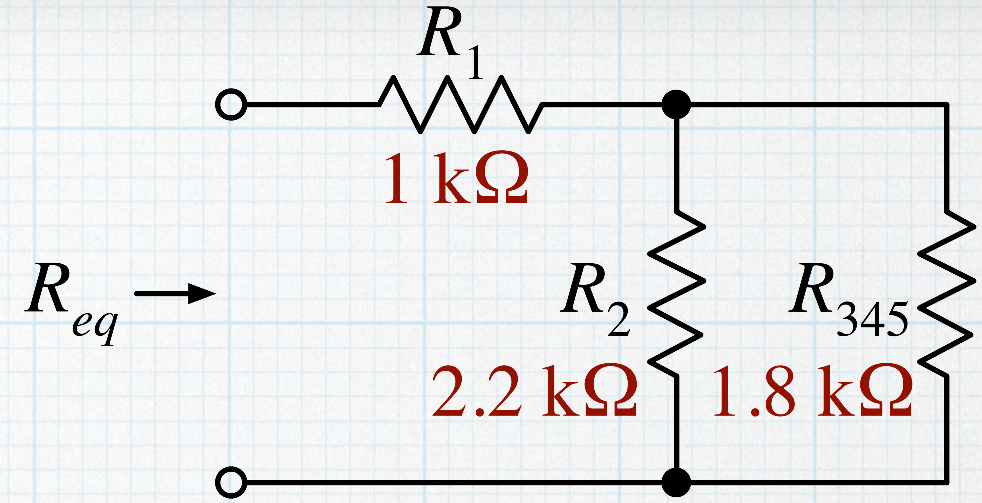


The trick is to break the circuit into smaller pieces that are purely series or parallel, find the equivalent of that piece and insert that back into the original circuit, which will now be simpler. Then find another series or parallel combination that can be simplified. Through a sequence of steps, it may be possible to reduce even complex combinations to a single equivalent resistance.

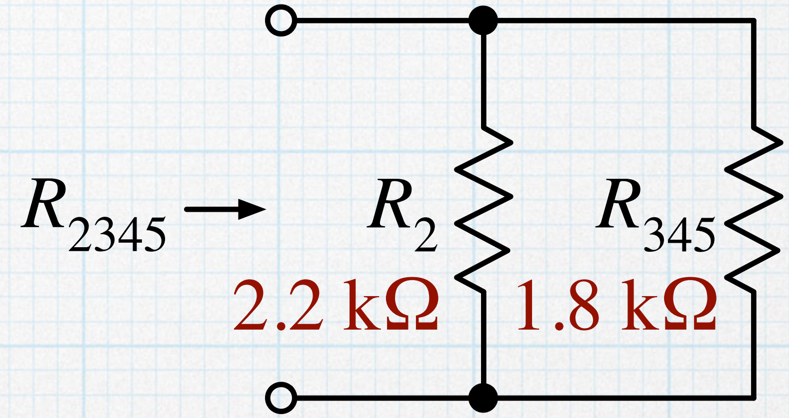
For the circuit above, we can start by recognizing that the R_3 - R_4 - R_5 series combination can be reduced:
$R_{345} = R_3 + R_4 + R_5 = 1.8\ \text{k}\Omega$.



Insert the single R_{345} resistor back into the original circuit. Now, quite obviously, R_2 is in parallel with R_{345}.

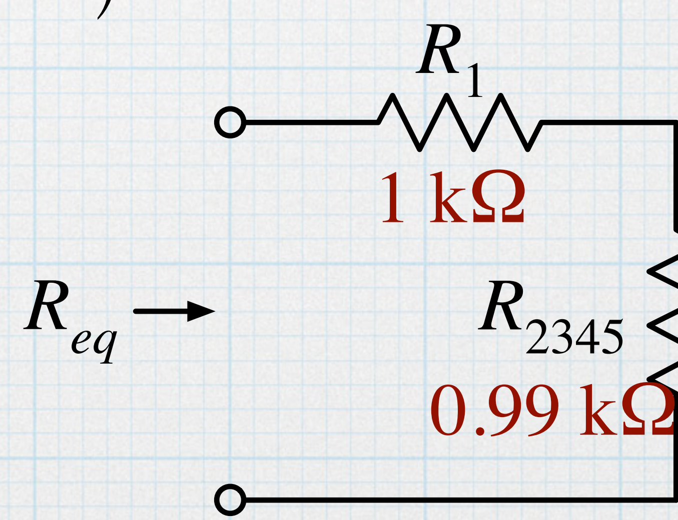


Calculate the equivalent resistance of the parallel combination. Using the two-resistor formula:



$$R_{2345} = \frac{R_2 \cdot R_{345}}{R_2 + R_{345}} = \frac{(2.2\,\mathrm{k\Omega})\,(1.8\,\mathrm{k\Omega})}{(2.2\,\mathrm{k\Omega}) + (1.8\,\mathrm{k\Omega})} = 990\,\Omega$$

Insert the R_{2345} equivalent back into what is left of the original circuit. Now, we easily calculate R_{eq} as the series combined of R_1 and R_{2345}.



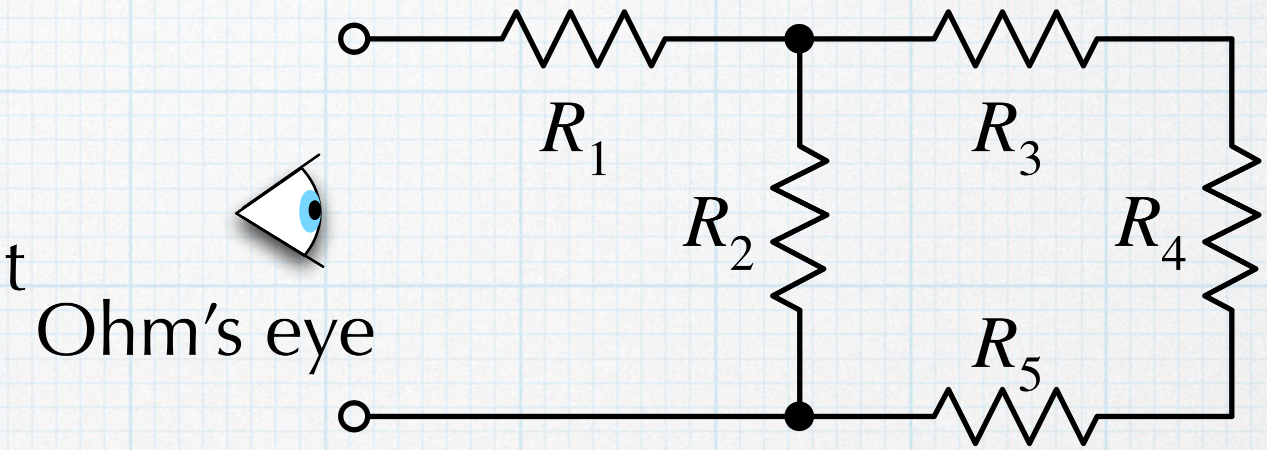
$$R_{eq} = R_1 + R_{2345} = 1\ \mathrm{k\Omega} + 0.99\ \mathrm{k\Omega} = 1.99\ \mathrm{k\Omega}.$$

Calculating source power is now trivial.

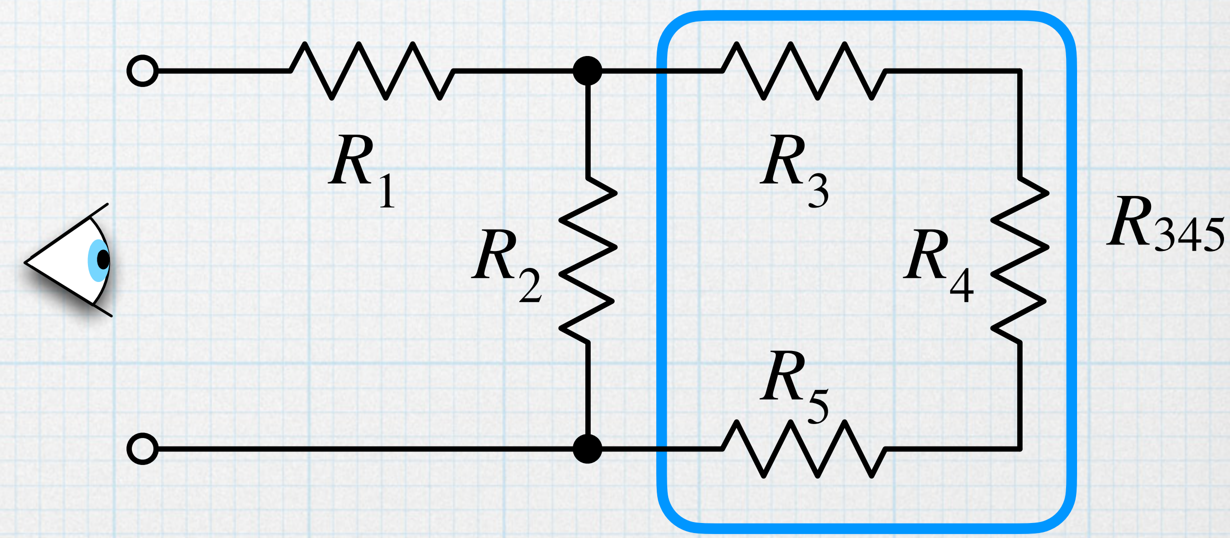
The equivalent resistance "method"

So we have a method for trying to find equivalent resistances without having to resort to messy combinations of Kirchoff's Laws.

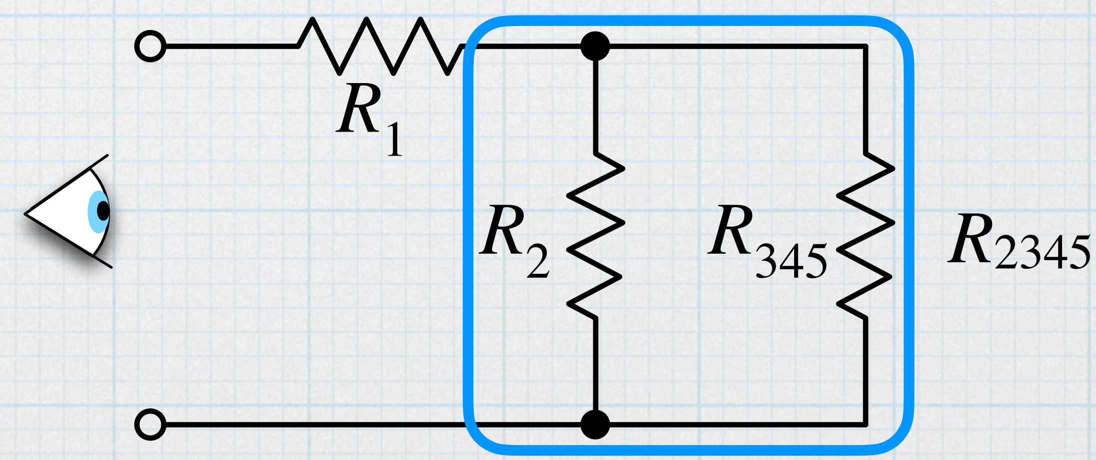
1. Identify the pair of nodes between which we want to find equivalent resistance. Peer into it with "Ohm's eye".



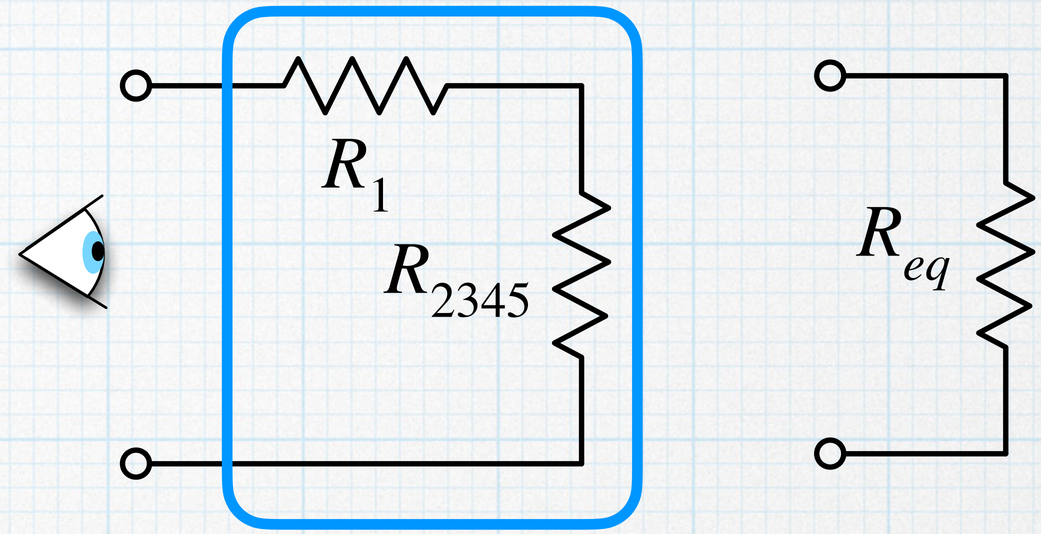
2. Starting at the opposite end of the network, identify series and parallel combinations that can be reduced using the simple formulas.



3. Repeat with another series or parallel combination to further simplify the circuit.



4. Continue the simplification process, one series or parallel combination at a time, until the network is reduced to a single resistor. (Or until the remaining network is trivial.)



It is not necessary to insert numbers at each step — we could express the results using symbols and then insert numbers at the end, if needed. For the example, the equivalent resistance expressed in symbols:

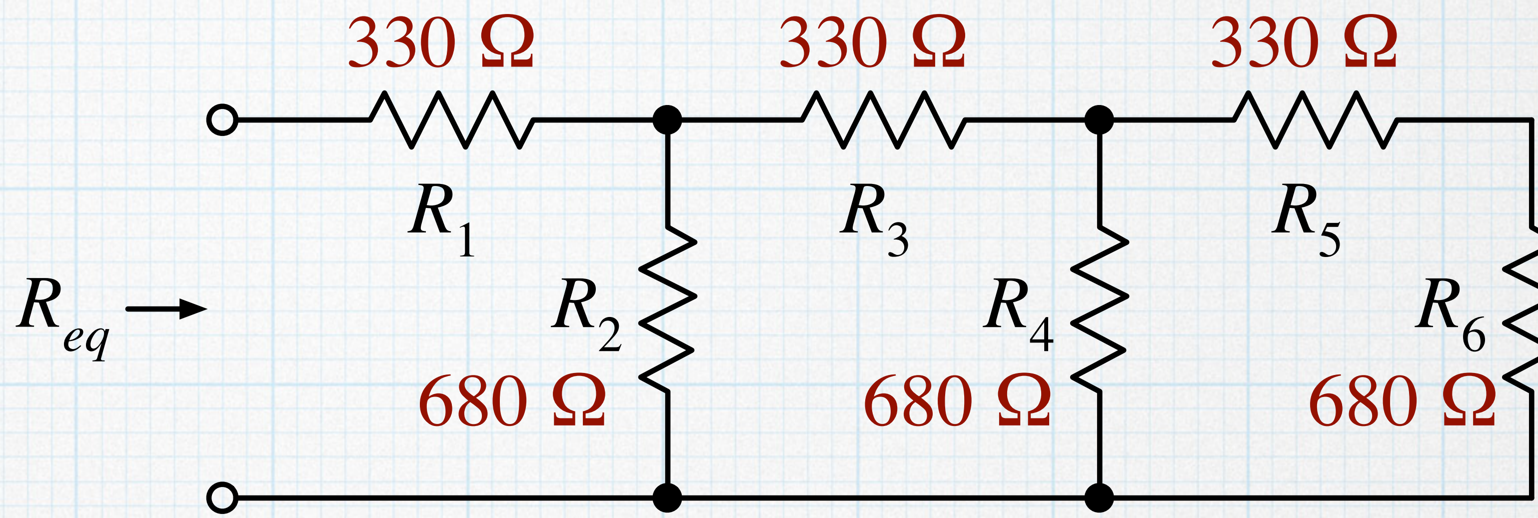
$$R_{eq} = R_1 + R_{2345} = R_1 + \frac{R_2 \cdot R_{345}}{R_2 + R_{345}} = R_1 + \frac{R_2 \cdot \left(R_3 + R_4 + R_5\right)}{R_2 + R_3 + R_4 + R_5}$$

With practice, many circuits can be simplified by inspection (i.e. in our heads). We might even be able to calculate the values in our heads.

Not all resistive networks can be reduced using series / parallel combinations. Consider the bridge circuit that was one of the Kirchoff's Laws practice problems — the bridging resistor is not in series or parallel with any other resistors and so there are no simplifications.

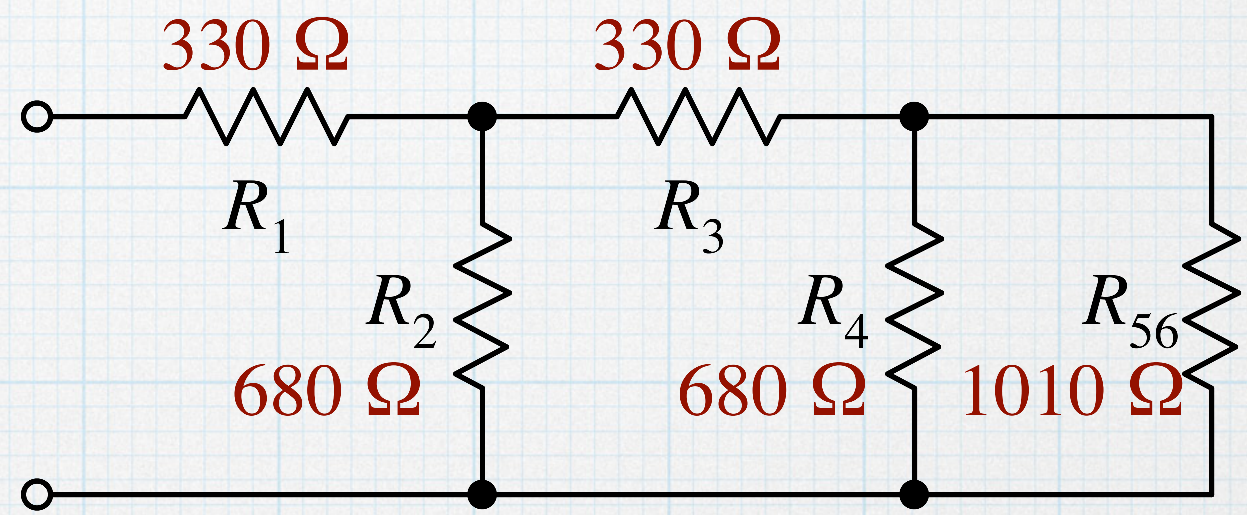
Example 1

Find the equivalent resistance looking into the indicated port of the "ladder network" shown.

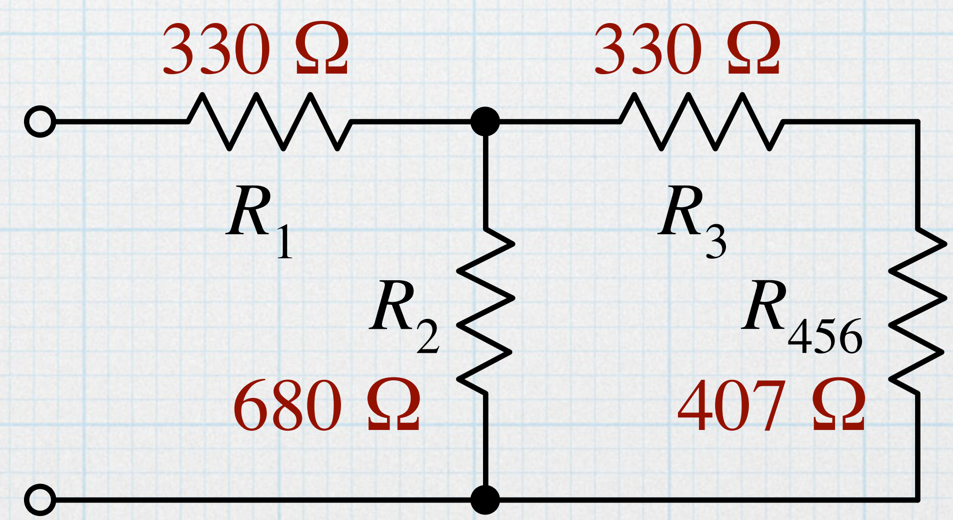


1. Starting at the "far end", we see that R_5 and R_6 are in series.

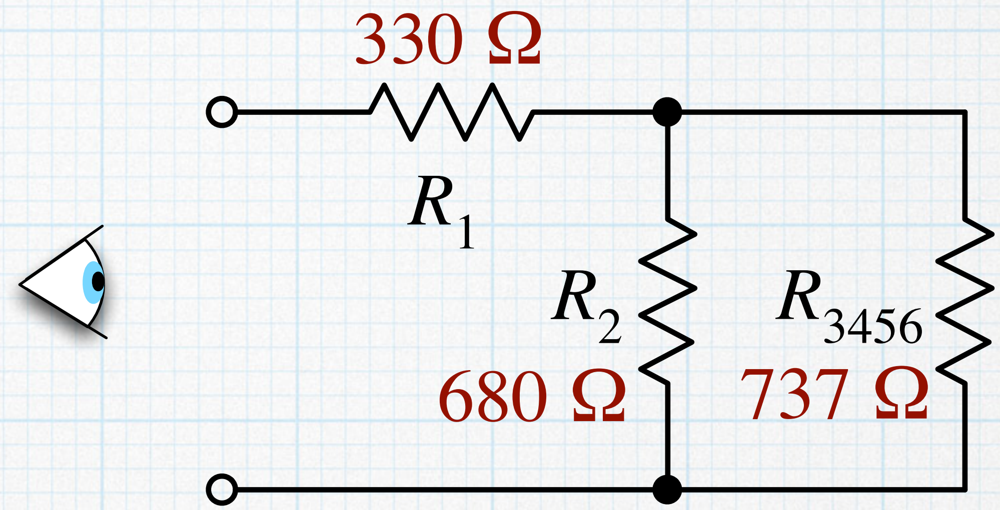
$$R_{56} = R_5 + R_6 = 1010\ \Omega.$$



2. R_4 is in parallel with R_{56}.
$R_{456} = (1/R_4 + 1/R_{56})^{-1} = 407\ \Omega.$

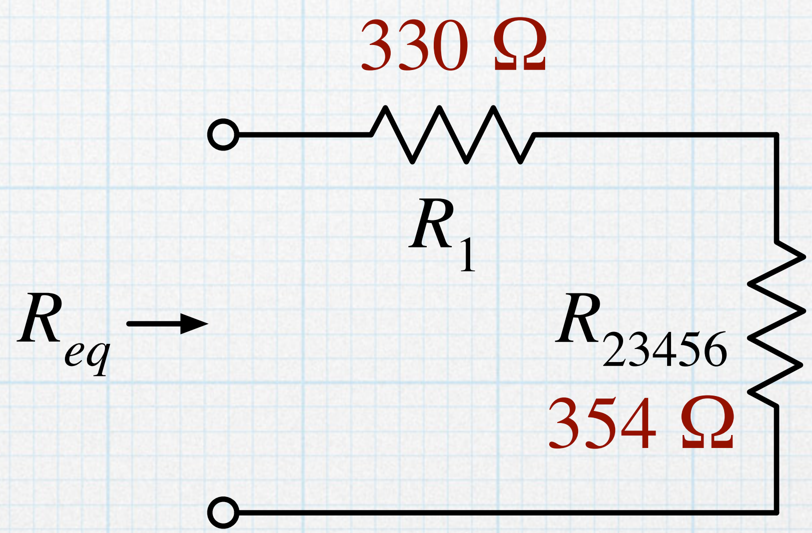


Example 1 (cont.)



3. R_3 and R_{456} are in series.

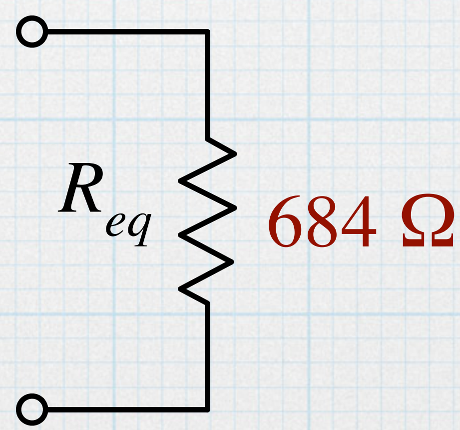
$R_{3456} = R_3 + R_{456} = 1010\ \Omega.$



4. R_2 is in parallel with R_{3456}.

$$R_{23456} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_{3456}}} = 354\,\Omega$$

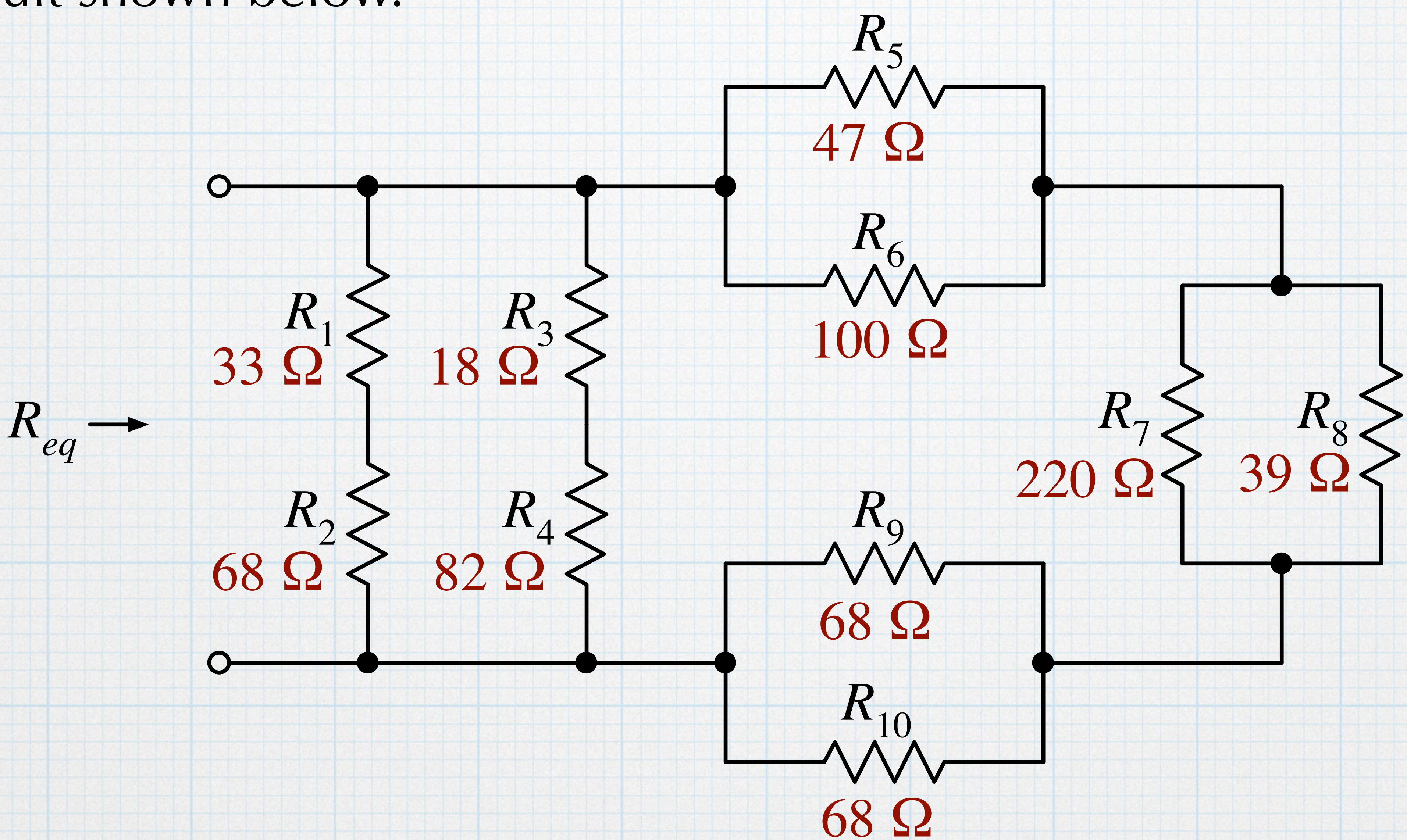
(OK — the subscripting thing is getting a bit ridiculous.)



5. Finally, R_{eq} is the series combination of R_1 and R_{23456}.
$R_{eq} = 330\ \Omega + 354\ \Omega = 684\ \Omega.$

Example 2

Find the equivalent resistance looking into the indicated port of the circuit shown below.



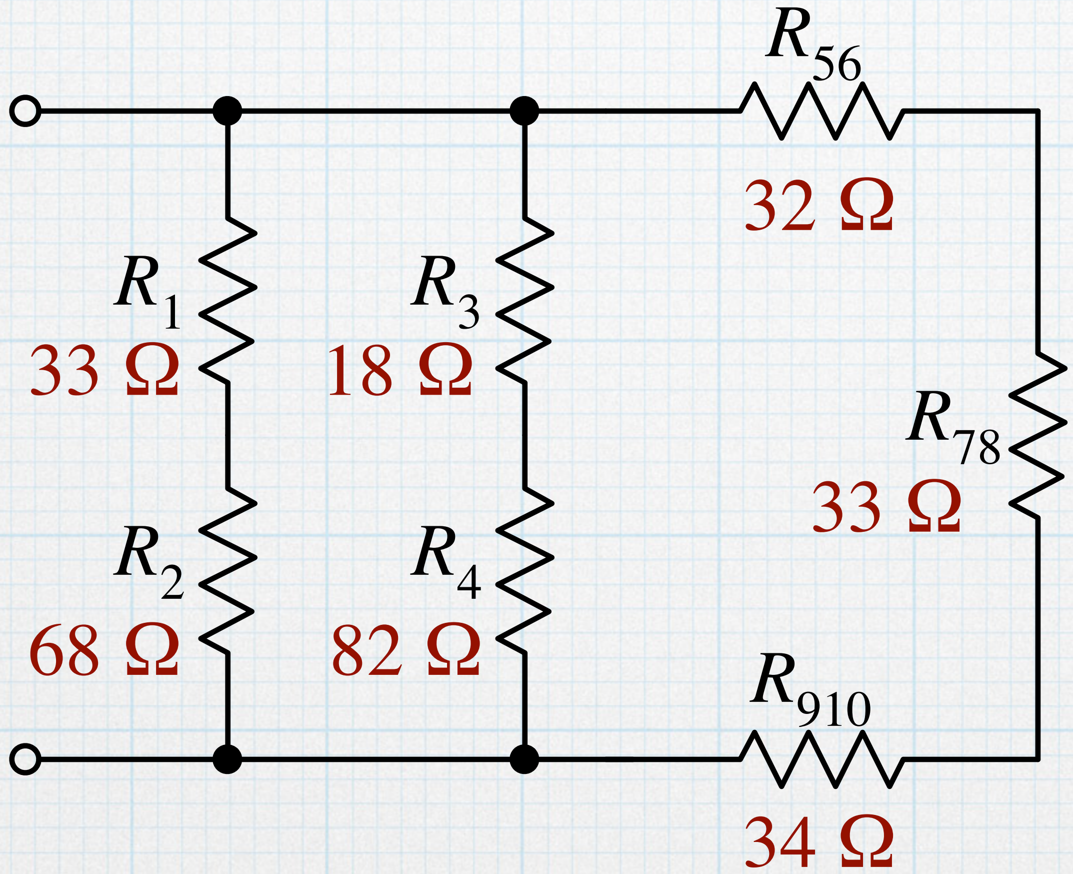
At first glance, this looks very difficult, but it's not so bad. We can pick it apart piece by piece. Start by noting that R_7 is in parallel with R_8.

$$R_{78} = \frac{1}{\frac{1}{R_7} + \frac{1}{R_8}} = 33.1\ \Omega$$

Example 2 (cont.)

Similarly, R_5 is in parallel with R_6 and R_9 is in parallel with R_{10}.

$$R_{56} = \frac{1}{\frac{1}{R_5} + \frac{1}{R_6}} = 32.0\,\Omega \qquad R_{910} = \frac{1}{\frac{1}{R_9} + \frac{1}{R_{10}}} = 34\,\Omega$$



Next, we note that there are several series combinations

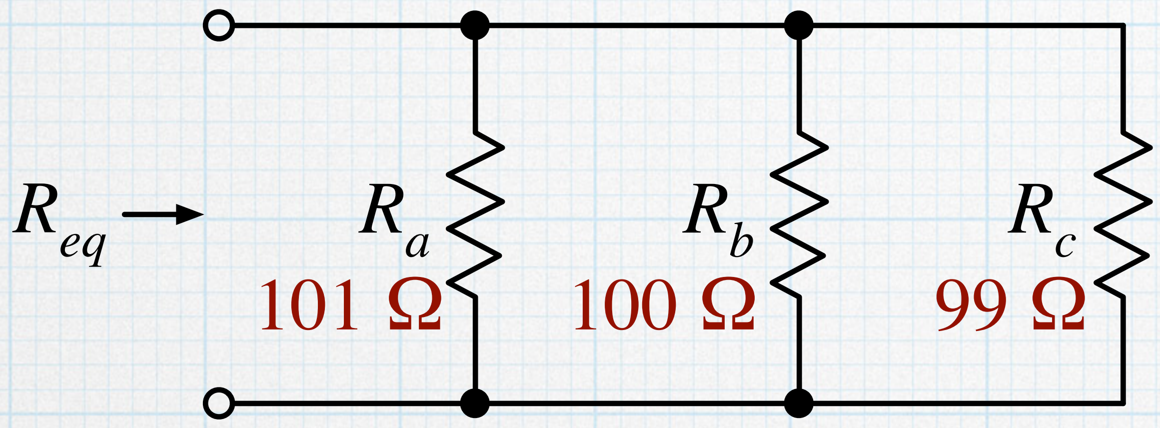
R_1 in series with R_2: $R_a = R_1 + R_2 = 101\,\Omega$

R_3 in series with R_4: $R_b = R_3 + R_4 = 100\,\Omega$

R_{56}, R_{78}, and R_{910} all in series: $R_c = R_{56} + R_{78} + R_{910} = 99\,\Omega$

Example 2 (cont.)

Finally, we see that the equivalent resistance is just the parallel combination of R_a, R_b, and R_c.

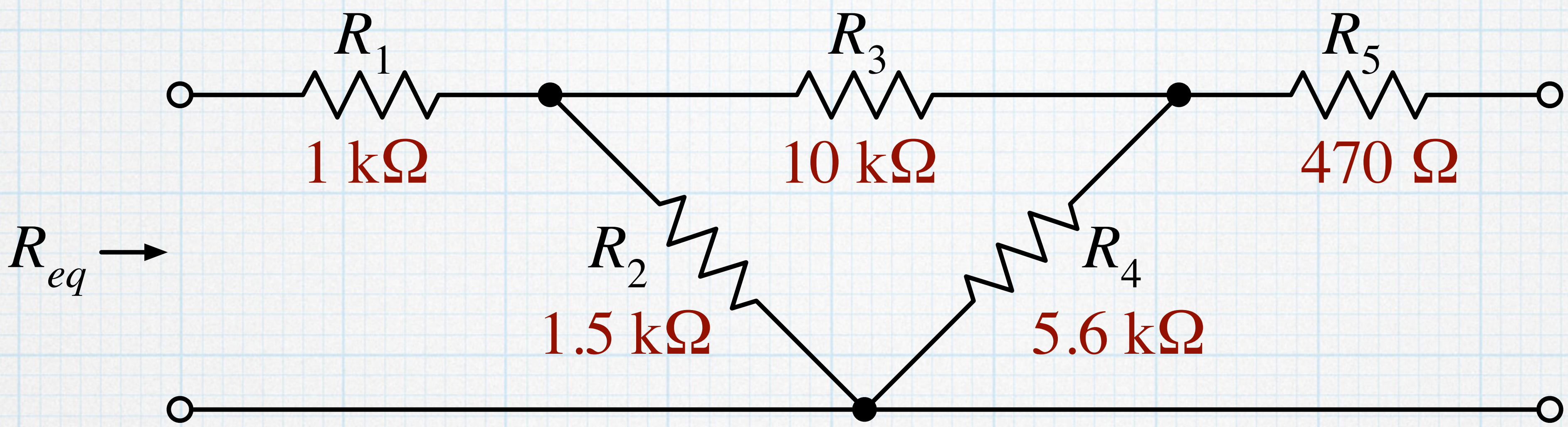


$$R_{eq} = \frac{1}{\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}} = 33.3\,\Omega$$

Not that bad.

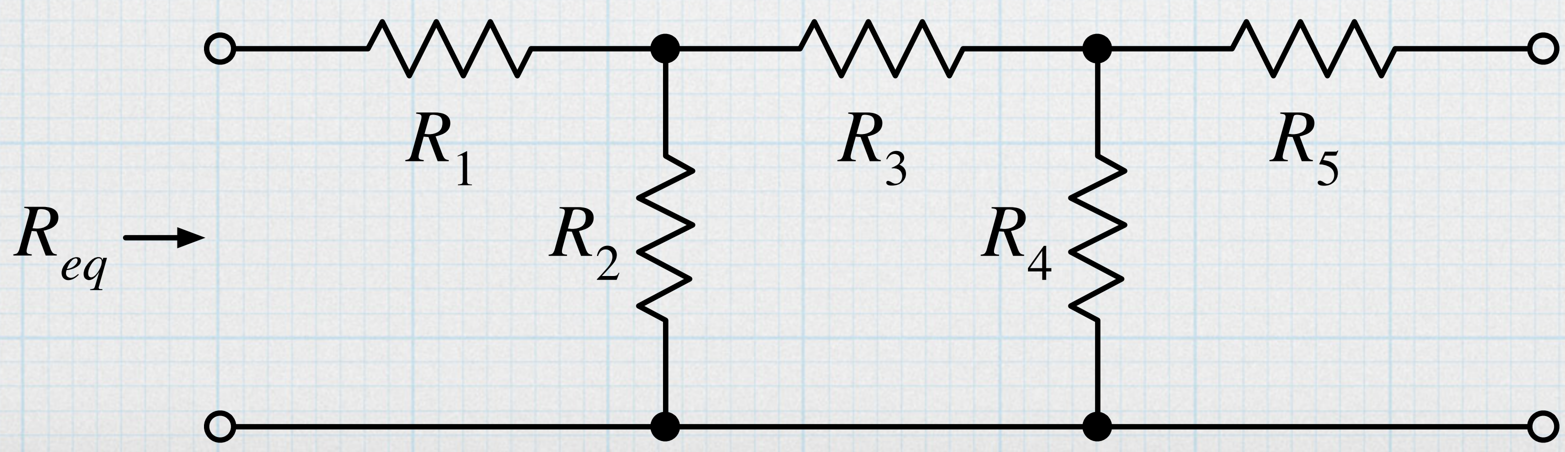
Example 3

Find the equivalent resistance at the indicated port in the circuit below.



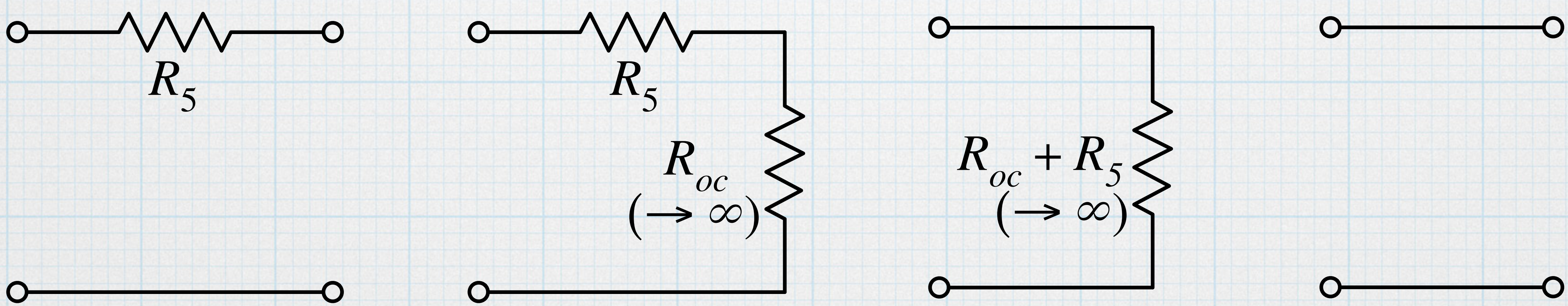
There are a couple of interesting things going on here. First, we see some "diagonal" resistors. Secondly, we see a "dangling" resistor, R_5, which is not connected to anything on one side.

First, the diagonal resistors are essentially an optical illusion — current and voltage do not care about the spatial orientation of the components. We can re-draw the circuit in the more familiar grid-like arrangement, with no change in how the circuit behaves.



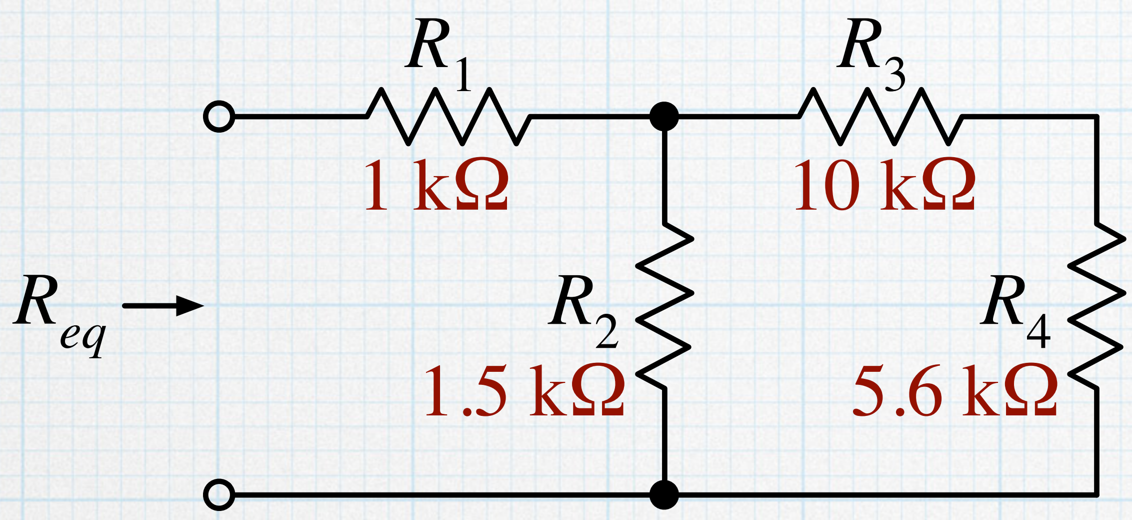
Example 3 (cont.)

Now, about the dangling resistor. Since the right-hand side of R_5 is "open circuited", we can view R_5 as being in series with a resistor with value approaching infinity. (An open circuit is essentially a resistor with $R \to \infty$.) A series combination of any finite resistor infinity is also infinity. (Mathematicians are cringing now.) So essentially, the dangling R_5 is the same as an open circuit — in principle, we could have left it off entirely with no change in equivalent resistance. (In the future, we will see a number of situations where there are dangling components like this, and we need to know how to handle them.)



Example 3 (cont.)

Now that we straightened out the diagonals and trimmed off the dangler, the circuit looks familiar and simple



And the calculation is straight-forward:

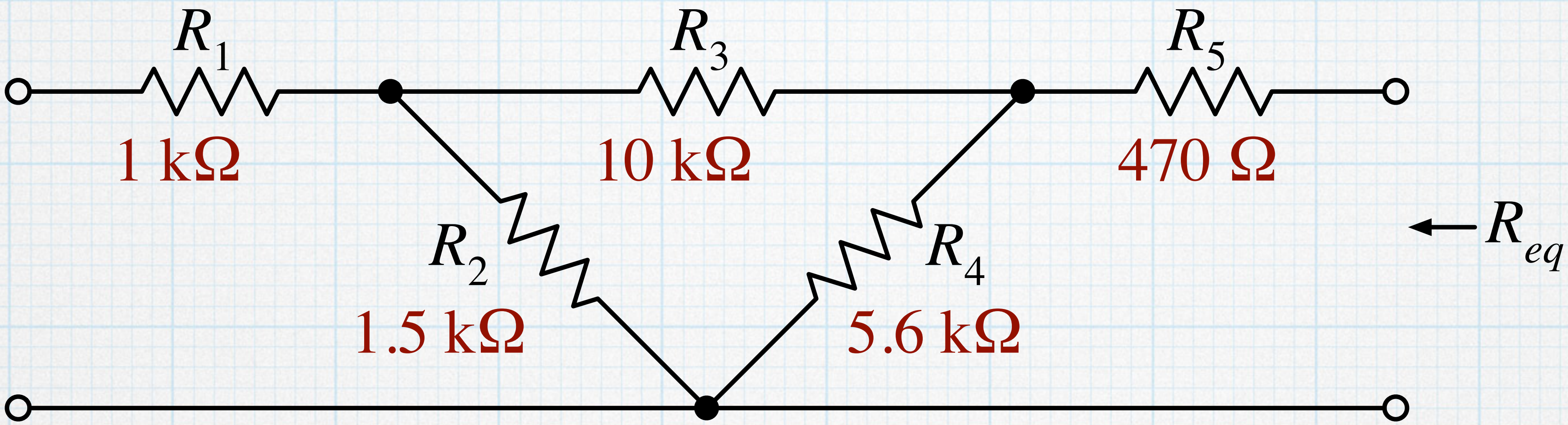
$$R_{34} = R_3 + R_4 = 15.6\ \text{k}\Omega$$

$$R_{234} = \frac{1}{\frac{1}{R_2} + \frac{1}{R_{34}}} = 1.37\ \text{k}\Omega$$

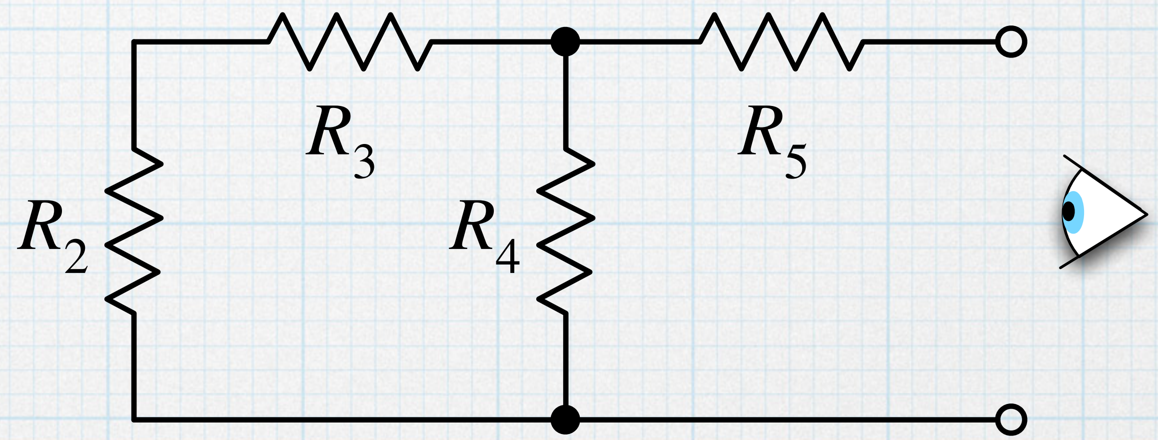
$$R_{eq} = R_1 + R_{234} = 2.37\ \text{k}\Omega$$

Example 3a

Same circuit, but now find the equivalent resistance looking from the other end.



The previous comments about the diagonals and the dangling resistor apply, except that now R_1 is the dangler.



We start at the "far end" and work towards the eyeball.

$$R_{23} = R_2 + R_3 = 11.5\ \text{k}\Omega$$

$$R_{234} = \frac{1}{\frac{1}{R_{23}} + \frac{1}{R_4}} = 3.77\ \text{k}\Omega$$

$$R_{eq} = R_5 + R_{234} = 4.24\ \text{k}\Omega$$

Voltage sources in series.

Consider the simple series circuit at right. We can write a KVL equation around the loop:

$$V_{S1} - v_{R1} - V_{S2} - v_{R2} = 0.$$

Addition and subtraction are commutative, so we can re-arrange the ordering in the equation.

$$V_{S1} - V_{S2} - v_{R1} - v_{R2} = 0.$$

This would imply that we can re-order the components in the circuit. The re-ordered circuit is must behave the same as the top circuit.

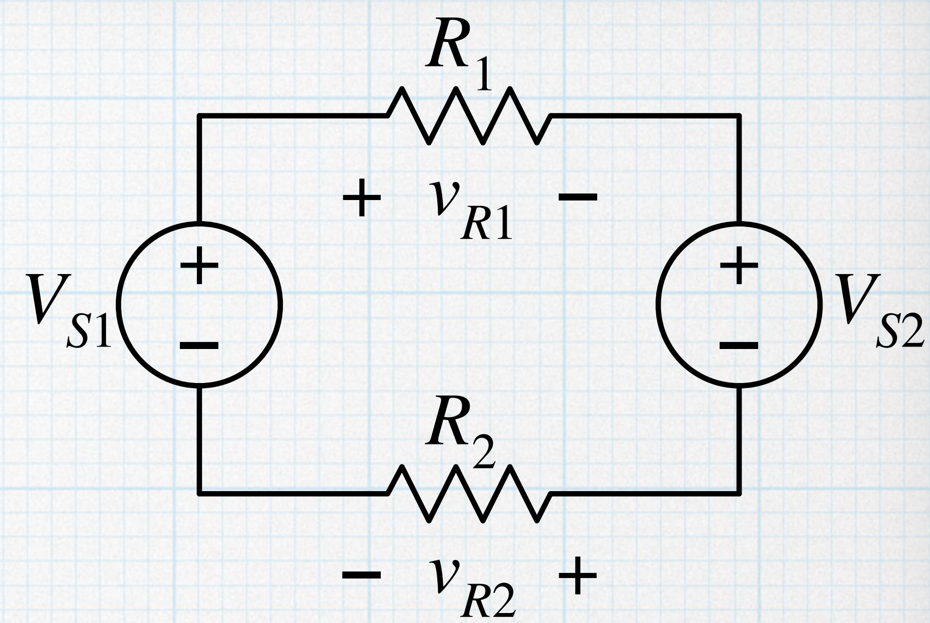
Now we can use Ohm's Law to write

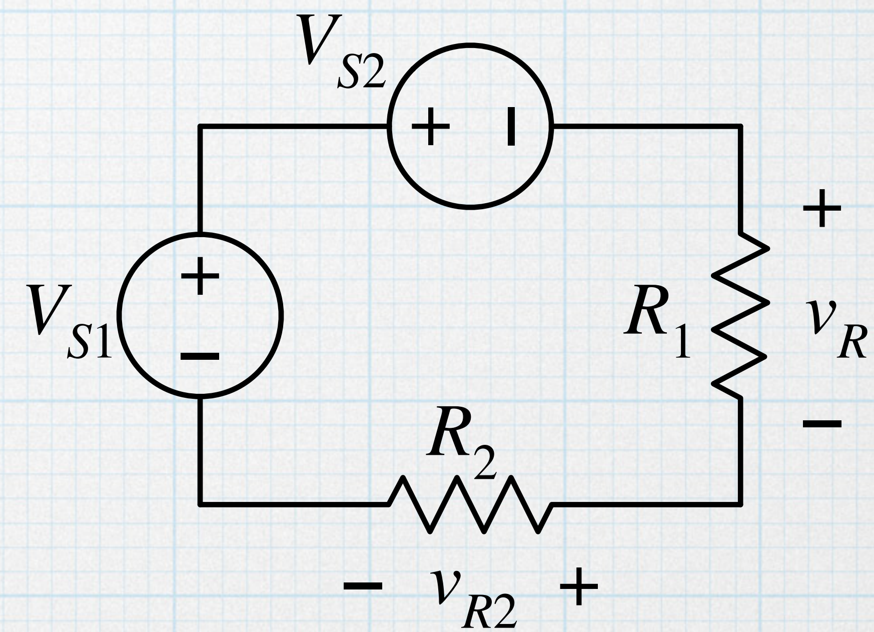
$$V_{S1} - V_{S2} - i_{R1} \cdot R_1 - i_{R2} \cdot R_2 = 0.$$

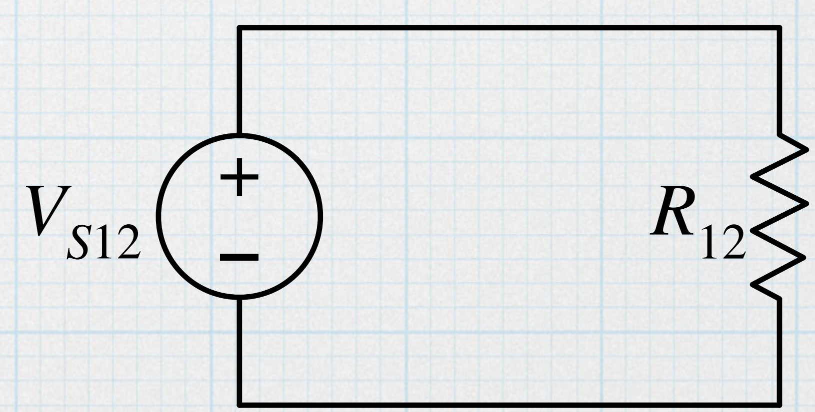
Since the same current flows in all components in the series string, $i_{S1} = i_{S2} = i_{R1} = i_{R2} = i_S$.

$$V_{S1} - V_{S2} - i_S (R_1 + R_2) = 0.$$

We know that we can combine series resistors. It appears that we can also combine series voltage sources: $V_{S12} - i_S R_{12} = 0$.







$V_{S12} = V_{S1} - V_{S2}$

$R_{12} = R_1 + R_2$

Voltage sources in series.

The little exercise on the previous slide show us important ideas about series connections.

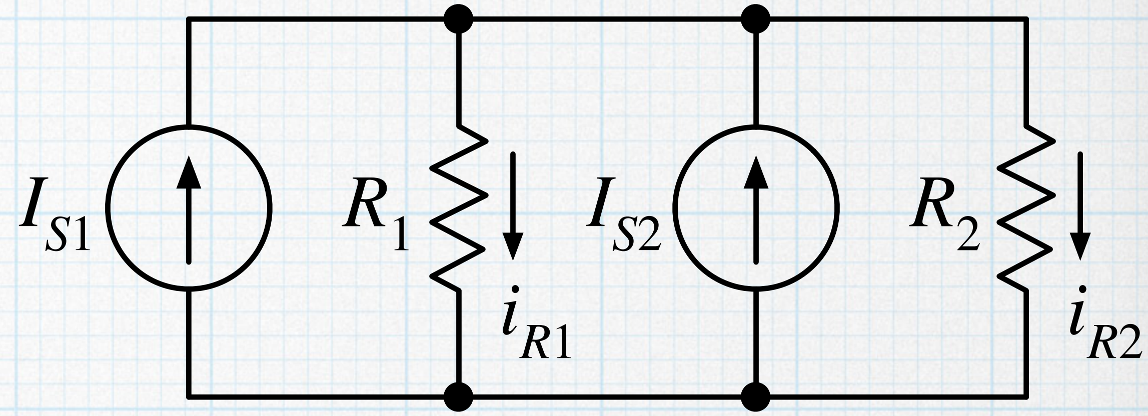
1. The ordering of components in the series string is irrelevant — we can re-order the voltage sources and resistors to suit our needs.
2. Just like resistors in series, we can combine voltage sources in series and treat them as a single source.

The idea of putting voltage sources in series should be familiar to most — in electronic gadgets it is common to connect several 1.5-V batteries in series to create 3-V or 4.5-V or 6-V or whatever voltage is needed to power a circuit.

When combining series voltage sources, there might some uncertainty about whether to add or subtract the values (particularly for neophytes). The ambiguity can always removed by writing a proper KVL equation around the loop. Kirchoff will make it clear whether to add to or subtract.

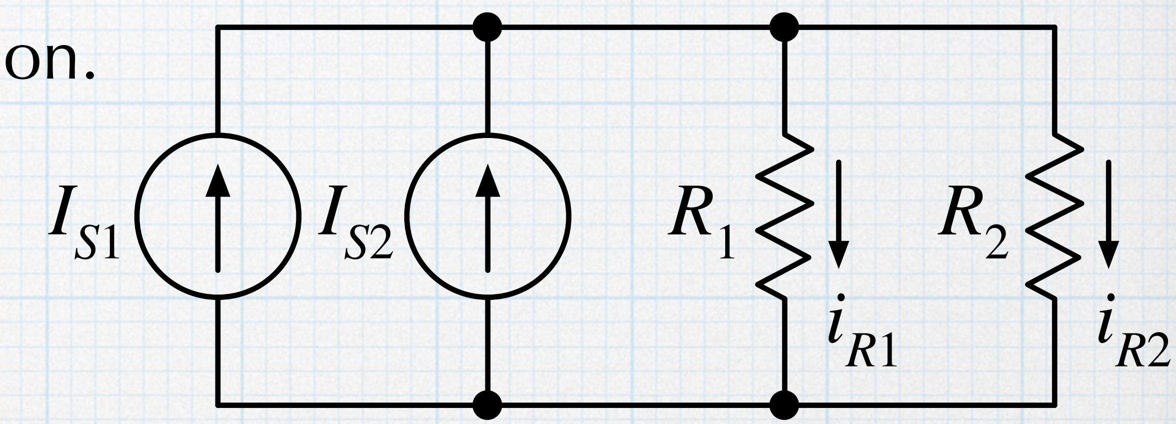
Current sources in parallel.

Consider the simple circuit at right. We can write a KCL equation at the top node:



$$I_{S1} - i_{R1} + I_{S2} - i_{R2} = 0.$$

Addition and subtraction are commutative, so we can re-arrange the ordering in the equation.

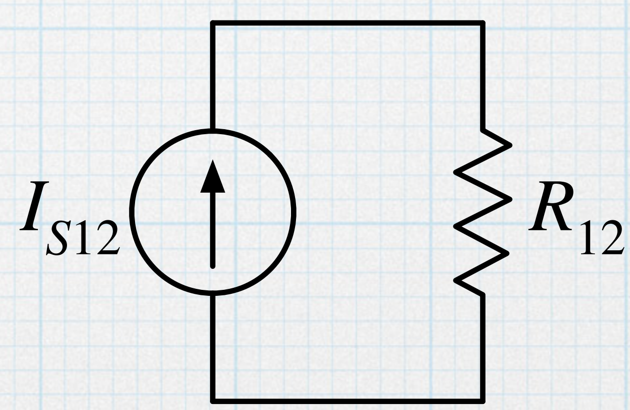


$$I_{S1} + I_{S2} - i_{R1} - i_{R2} = 0.$$

This would imply that we can re-order the components in the circuit. The re-ordered circuit must be identical to the top circuit.

We can use Ohm's Law to write

$$I_{S1} + I_{S2} - \frac{v_{R1}}{R_1} + \frac{v_{R2}}{R_2} = 0$$



All the components have the same voltage across, $v_{IS1} = v_{IS2} = v_{R2} = v_{R2} = v_S$, We know that we can combine the parallel resistors, and it appears that we can combine the current sources as well.

$$I_{S1} + I_{S2} - \frac{v_S}{R_1} - \frac{v_S}{R_2} = 0 \;\rightarrow\; I_{S12} - \frac{v_S}{R_{12}} = 0 \qquad I_{S12} = I_{S1} + I_{S2} \qquad R_{12} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

Current sources in parallel.

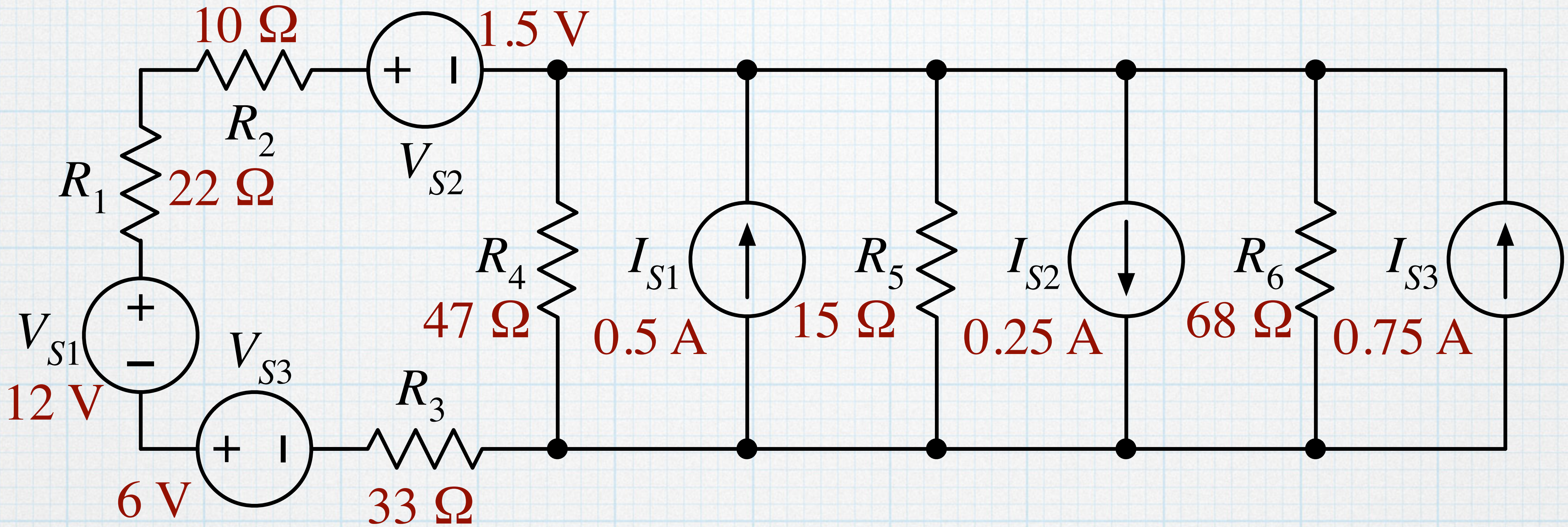
The little exercise on the previous slide show us to important ideas about parallel connections.

1. The ordering of components in the parallel arrangement is irrelevant — we can re-order the parallel current sources and resistors to suit our needs.
2. We can combine current sources in parallel and treat them as a single source.

When combining parallel current sources, there is often some uncertainty about whether to add or subtract the values. The ambiguity can always removed by writing a proper KCL equation at the node where they are connected. Kirchoff will make it clear whether to add to or subtract.

Example 4

Below is a conglomeration of sources and resistors. Simplify the circuit by combining the series and parallel components.



Three resistors in series on the left: $R_L = R_1 + R_2 + R_3 = 65\ \Omega$

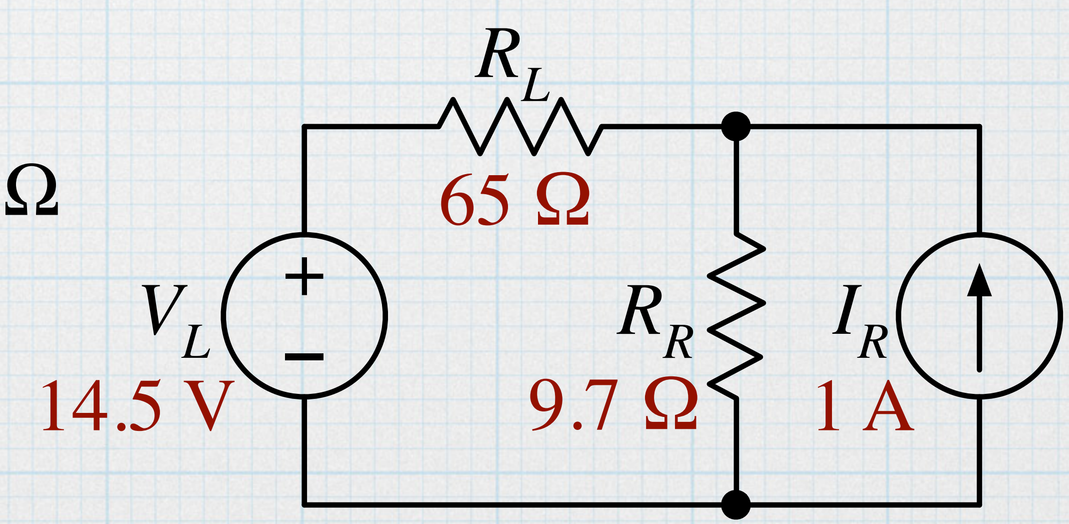
Three sources in series on the left: $V_L = V_{S3} + V_{S1} - V_{S2} = 14.5\ \mathrm{V}$

Three resistors in parallel on the right:

$$R_R = \left(R_4^{-1} + R_5^{-1} + R_6^{-1}\right)^{-1} = 9.74\,\Omega$$

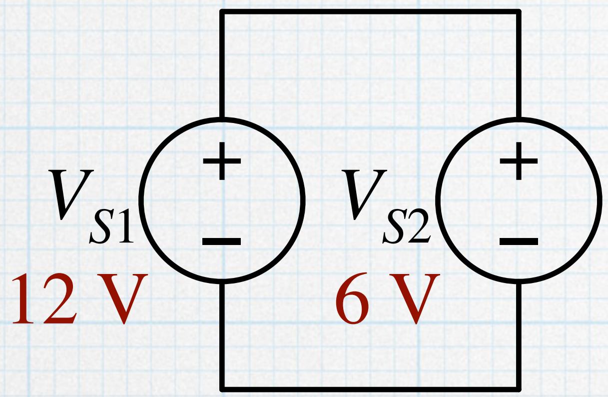
Three sources in parallel on the right:

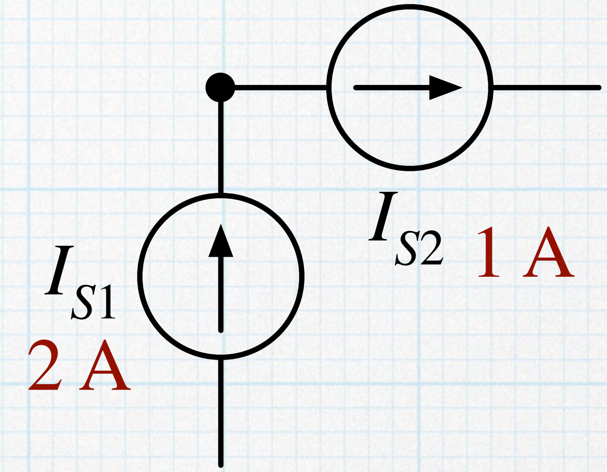
$$I_R = I_{S1} - I_{S2} + I_{S3} = 1\ \mathrm{A}.$$



Voltage sources in parallel, current sources in series

From a theoretical point of view, these combinations are not allowable. They lead to untenable conundrums with Kirchoff's Laws.

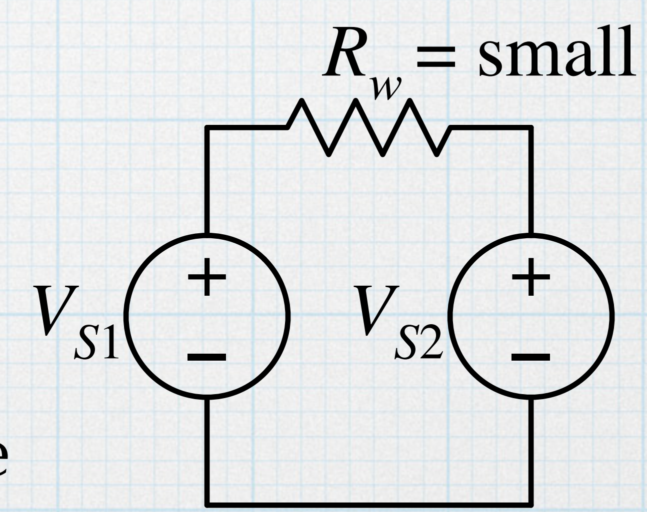




KVL: $V_{S1} - V_{S2} = 6\text{ V} \neq 0$!! Yikes!

KCL: $I_{S1} \neq I_{S2}$: In ≠ Out!! Yikes!

So in 201 circuits, we avoid these. However, everyone knows that sometimes voltage sources are connected to parallel — charging a battery is essentially requires connecting one source to another. If there were no other considerations, then the resistance of the wire (which we generally ignore in 201) comes into play.



$$i = \frac{V_{S1} - V_{S2}}{R_w} = \text{BIG!}$$

If we connect two random batteries together (or short out a battery — $V_{S2} = 0$), bad things may happen. A practical battery charger will have some means to limit current. In fact, it may actually be current source.