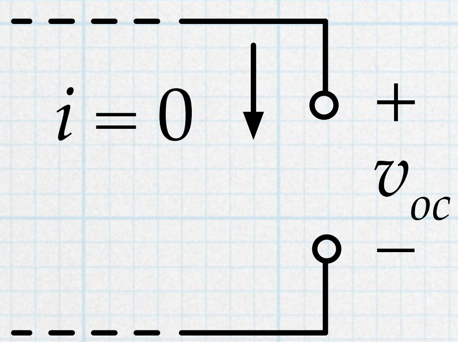
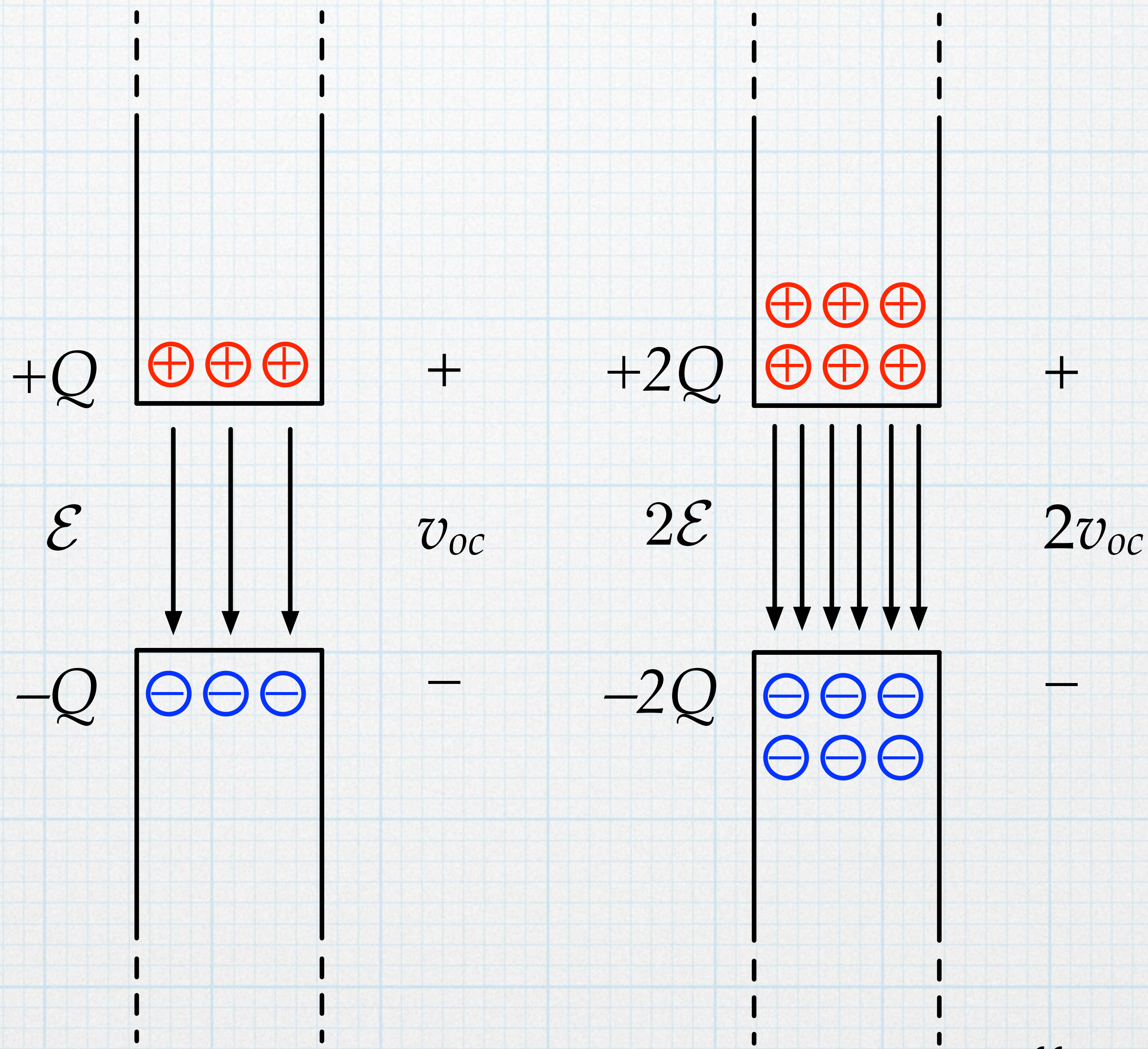
Capacitors

Consider an open circuit:



double the voltage



$Q \propto v_{oc}$

$\mathcal{E} \propto v_{oc}$

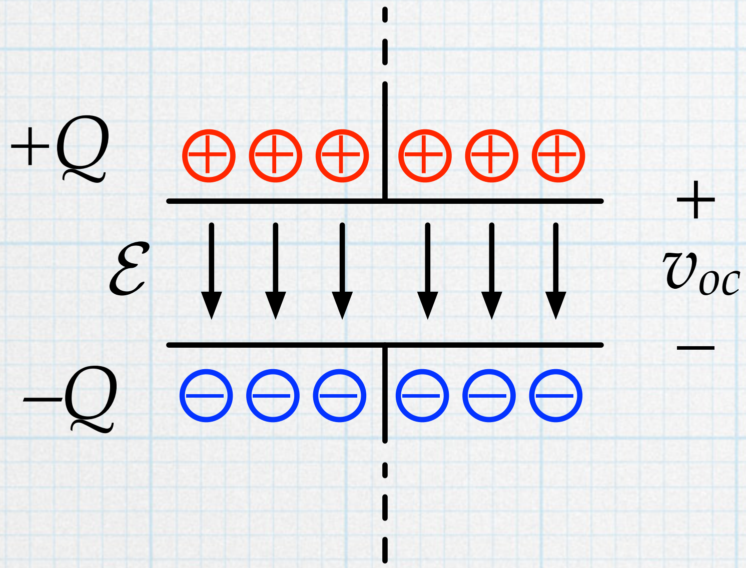
There is some energy "stored".

There was a bit of current flow when the voltage changed.

Effect is weak for dangling wires. (But not zero!)

Capacitance

Change the geometry – have parallel plates with area A



same voltage

- much more charge
- much more electric field
- much more energy stored

$Q = CV$

$Q = \epsilon \mathcal{E} A$

$\mathcal{E} = \frac{V}{d}$

C → capacitance

farads (F) = C / V

increase charge with better dielectric material and more area.

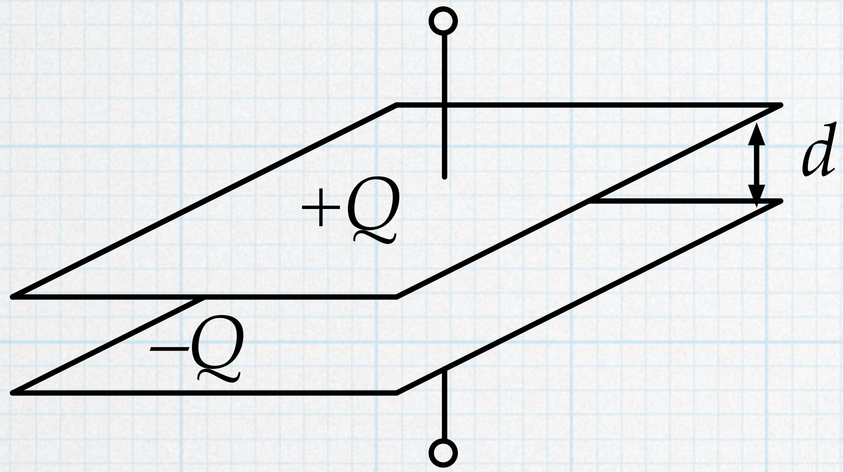
increase field (and hence Q) by moving plates closer together

air: ε_o = 8.85x10^{-12} F/m

other materials: $\varepsilon = \varepsilon_r \varepsilon_o$

relative dielectric: ε_r = constant

Parallel-plate capacitor



2 plates, each with area A.

$$C = \frac{\epsilon A}{d}$$

Example: $A = 1\ \text{cm}^2$, $d = 0.001\ \text{cm}$, air dielectric

$$C = \frac{\left(8.8 \times 10^{-14}\text{F}/\text{cm}\right)\left(1\text{cm}^2\right)}{0.001\text{cm}} = 8.85 \times 10^{-11}\text{F} = 88.5\text{pF}$$

Wow. Very small value, and it is already a fairly large area plate.

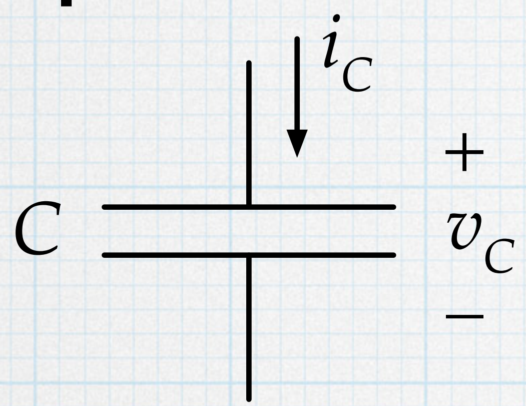
Higher values?

- higher dielectric material between the electrodes
- thinner dielectric
- winding or stacking to get larger surface area into a smaller volume.

Other configurations are possible, but parallel-plate is most common.

Values range from 10 pF to 100 µF, (and higher). A 1-F capacitor is huge and quite rare.

Capacitor current



Note that passive sign convention.

At DC, $i_c = 0$. (It's just a fancy open circuit.)

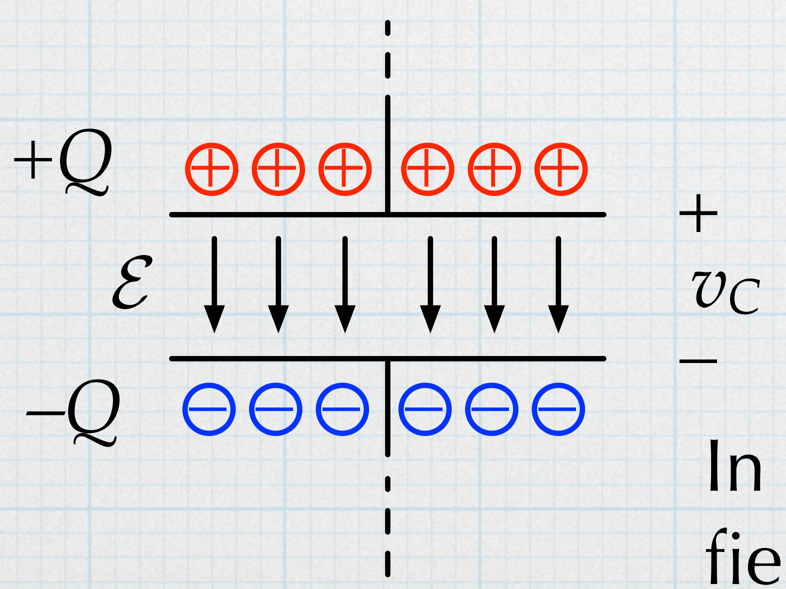
However, some current must flow when voltage is *changing*. Otherwise, the charge would not change.

$$Q = Cv_C$$

$$\frac{dQ}{dt} = i_C = C\frac{dv_c}{dt}$$

Current only flows when voltage is changing.

As current flows, the capacitor charge increases or decreases.

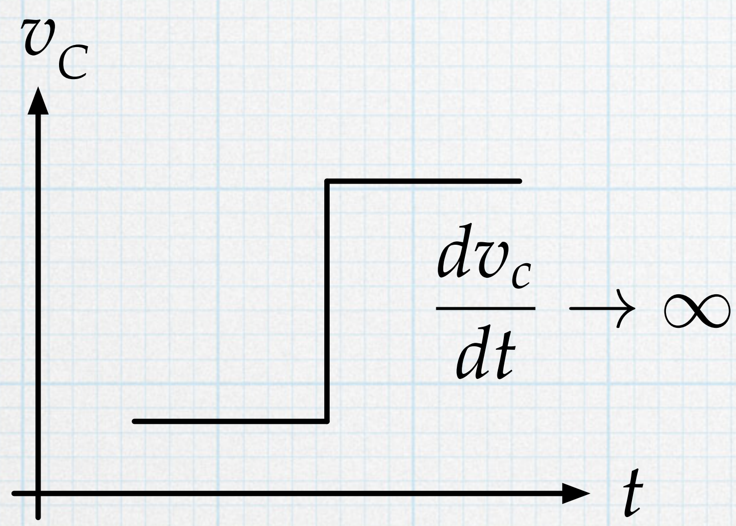


$$\frac{dv_C}{dt} \rightarrow \frac{d\mathcal{E}}{dt} \qquad \frac{d\mathcal{E}}{dt} \rightarrow \frac{dQ}{dt}$$

In Maxwell's equations, the current due to changing field is called *displacement* current.

$$i_C = C\frac{dv_c}{dt}$$

Capacitor voltage cannot change instantaneously.



infinite current!

can't happen.

Note, though, that current can change instantaneously.

also $$v_C(t) = \frac{1}{C}\int_0^t i_C(t')\,dt' + v_C(0)$$

Capacitor energy

An energy storage device

- Charge the cap to some voltage. Charge (and energy) stays. Remove it later.
- Ideal capacitor dissipates no energy – no heat generated.
- Real capacitors do show some leakage. (Large resistor in parallel.) Usually negligible.

When charging a capacitor, the power being delivered is given by:

$$P_C(t) = v_C(t)\, i_C(t) = C v_C \frac{dv_C}{dt}$$

The energy delivered by the source, and hence the energy stored in the capacitor is (assuming $v_C = 0$ at $t = 0$ and $v_C(t_f) = V_C$.)

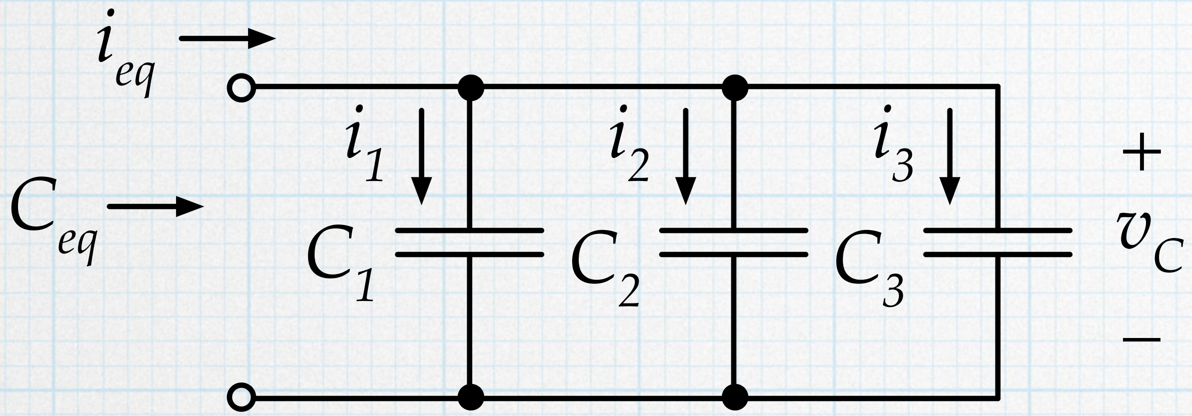
$$P_C(t)\, dt = C v_C dv_C$$

$$E = \int_0^{t_f} P_C(t)\, dt = C \int_0^{V_C} v_C dv_C$$

$$\boxed{E = \frac{1}{2} C V_C^2}$$

Combinations of capacitors

Parallel

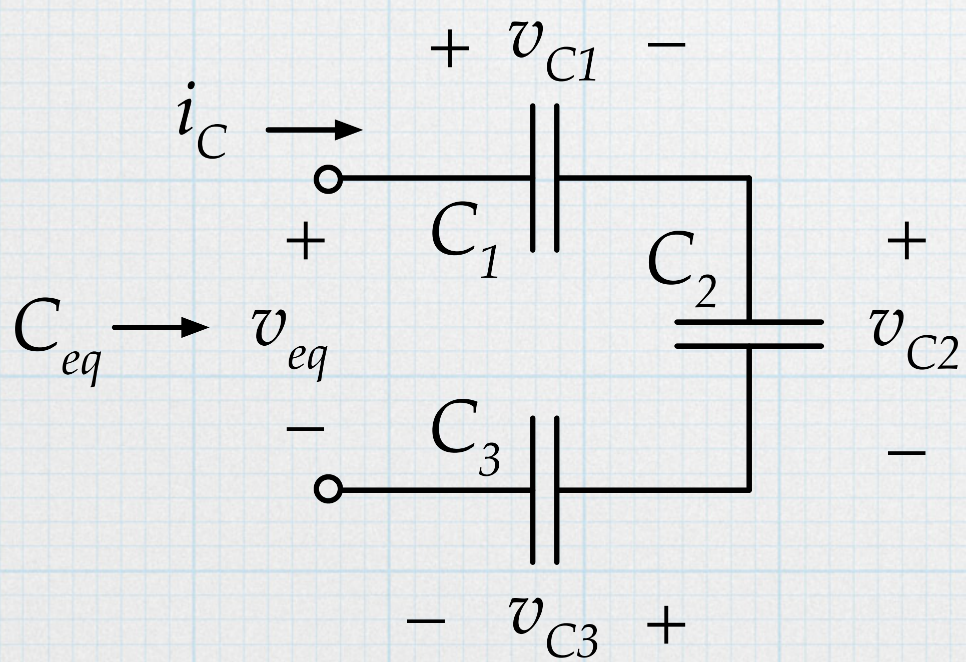


$$C_{eq} = C_1 + C_2 + C_3$$

$$i_{eq} = i_1 + i_2 + i_3$$

$$C_{eq}\frac{dv_C}{dt} = C_1\frac{dv_C}{dt} + C_2\frac{dv_C}{dt} + C_3\frac{dv_C}{dt}$$

Series



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$v_{eq} = v_{c1} + v_{c2} + v_{c3}$$

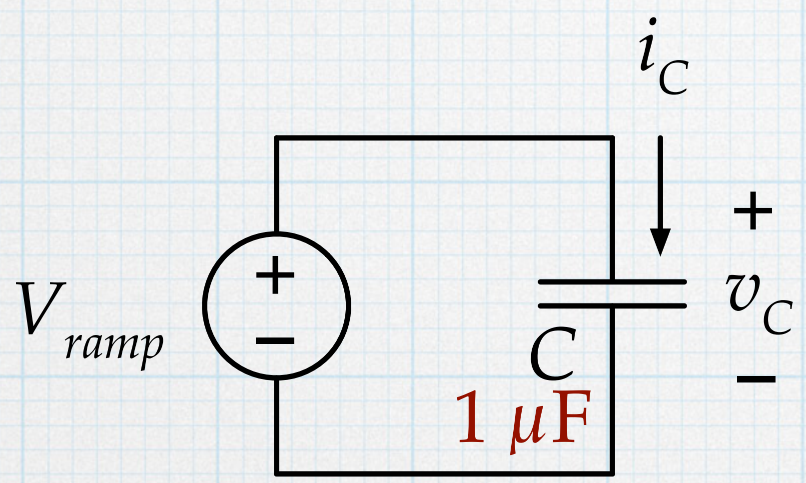
$$\frac{dv_{eq}}{dt} = \frac{dv_{c1}}{dt} + \frac{dv_{c2}}{dt} + \frac{dv_{c3}}{dt}$$

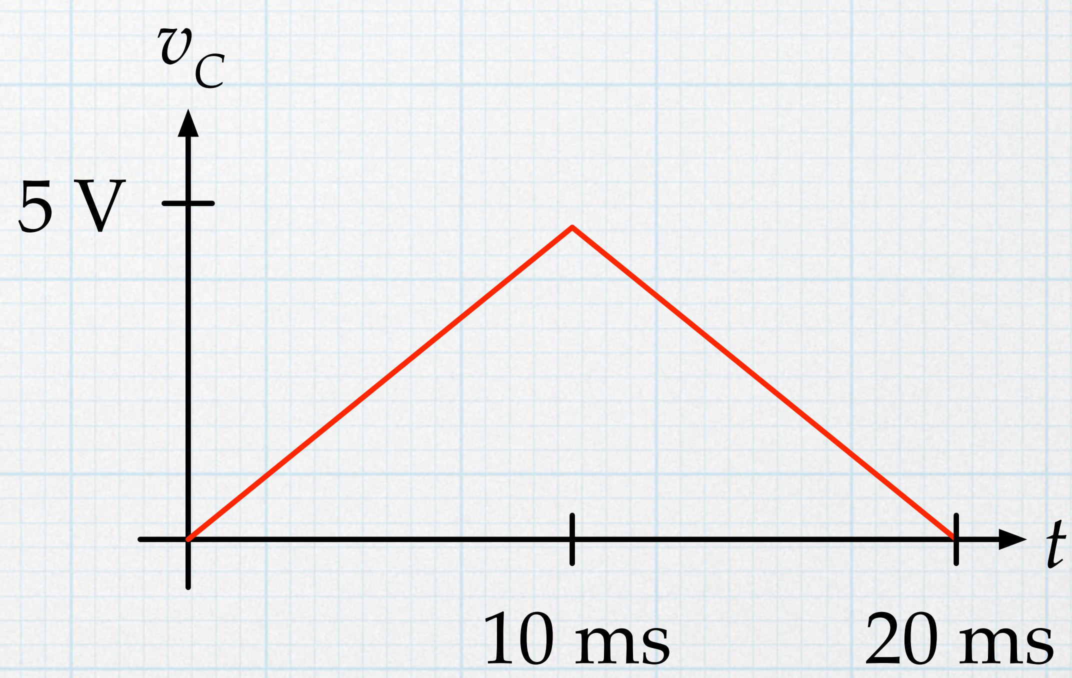
$$\frac{i_C}{C_{eq}} = \frac{i_C}{C_1} + \frac{i_C}{C_2} + \frac{i_C}{C_3}$$

Capacitors combinations are exactly opposite those of resistors.

Example

A voltage source connected across a capacitor has a ramp (or triangle) shape as a function of time. It ramps from 0 V to 5 V in 10 ms and then ramps back down to 0 V in another 10 ms. What is the current in the capacitor?





We can write expressions for the voltage as a function of time.

$$0 < t < 10\text{ ms: } v_C(\text{t}) = (5\text{V}/10\text{ms})\cdot t = (500\text{ V/s})\cdot t$$

$$10\text{ ms} < t < 20\text{ ms: } v_C(\text{t}) = (-500\text{ V/s})\cdot(t - 10\text{ ms}) + 5\text{ V}$$

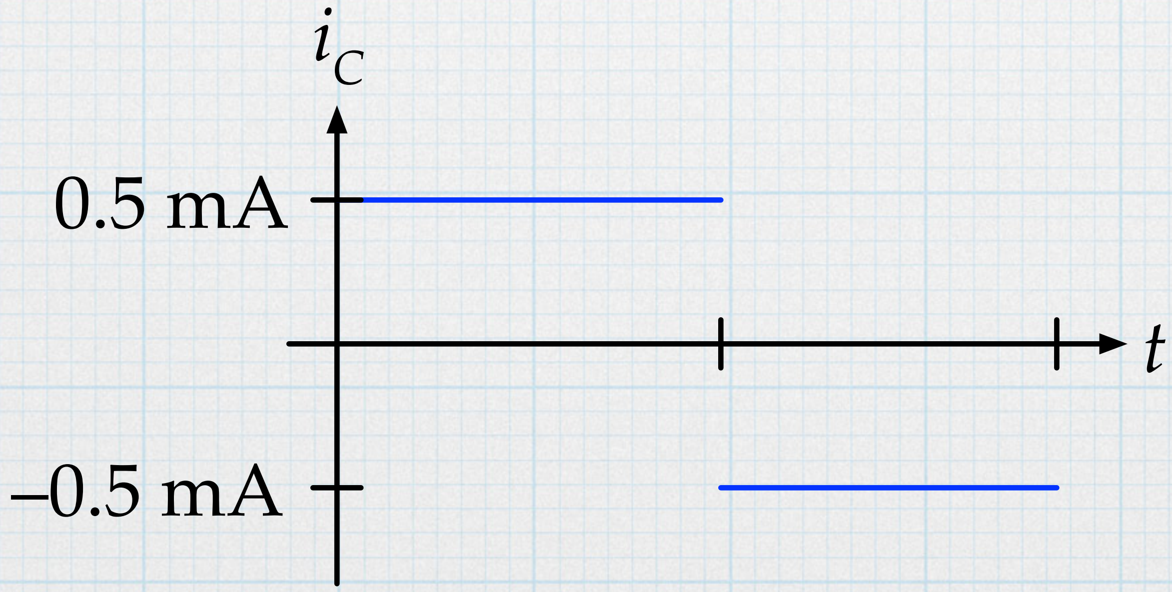
$$= -(500\text{ V/s})\cdot t + 10\text{ V}$$

We find the current by taking the derivative of the voltage and multiplying by the capacitance.

0 ms < t < 10 ms: $$i_C(t) = C\frac{dv_C}{dt} = (1\ \mu\text{F}) \cdot (500\ \text{V/s}) = +0.5\ \text{mA}$$

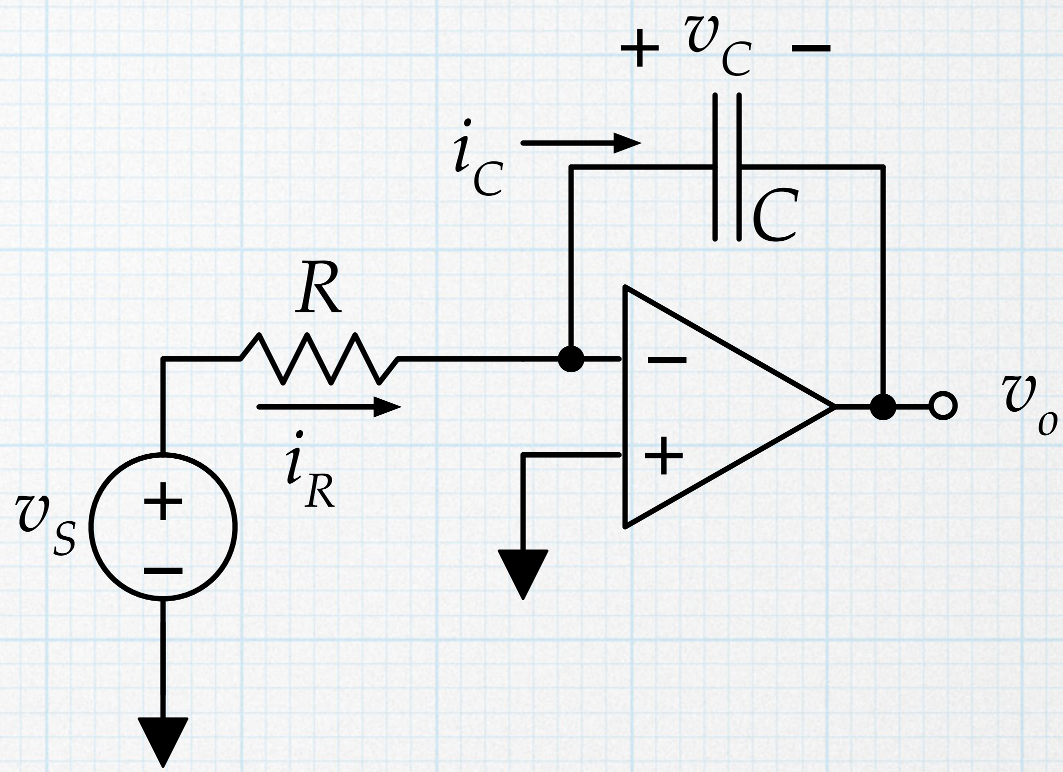
10 ms < t < 20 ms: $$i_C(t) = C\frac{dv_C}{dt} = (1\ \mu\text{F}) \cdot (-500\ \text{V/s}) = -0.5\ \text{mA}$$

The current is constant for each portion – positive current during the upward ramp and negative during the downward ramp.



Example

What is v_o for the circuit at right?



Start as always: write a node equation at the inverting input.

$$i_R = i_C + i_-$$

$$\frac{v_s - v_-}{R} = C\frac{dv_c}{dt} + i_-$$

For an op-amp with a feedback loop: $v_- = v_+$, so $v_- = 0$ in this case (virtual ground). And, for an ideal op amp: $i_- = 0$. Also, note $v_C = 0 - v_o$.

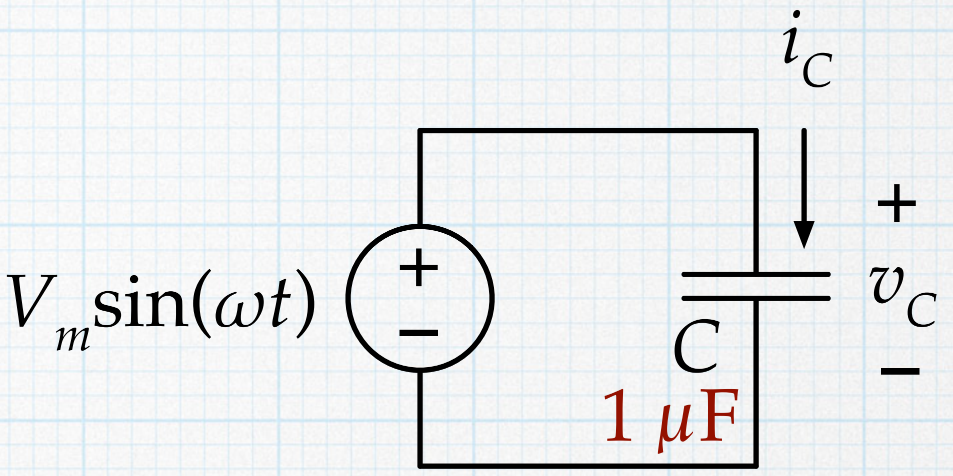
$$\frac{v_s}{R} = -C\frac{dv_o}{dt}$$

$$v_o(t) = \frac{1}{RC}\int_0^t v_s(t')\,dt' + v_o(0)$$

A circuit that integrates!

Example

The voltage across the capacitor at right is a sinusoid: $v_s(t) = V_m \sin(\omega t)$. What is the capacitor current?



$$i_C(t) = C\frac{dv_C(t)}{dt} = [\omega C V_m]\cos(\omega t)$$

$V_m = 5$ V; $\omega = 377$ rad/s

$$= I_m \cos(\omega t) = I_m \sin(\omega t - 90°) \quad I_m = 1.88 \text{ mA}$$

Interesting. The current has also a sinusoidal form, but it is shifted by 90°. Also, the magnitude of the current waveform depends on the frequency of the oscillation – faster oscillation leads to bigger currents. The frequency-dependent amplitude and the phase shift will have far-reaching implications when we study sinusoidal circuits in more detail.