AC analysis

Now we turn to circuits with sinusoidal sources. Earlier, we had a brief look at sinusoids, but now we will add in capacitors and inductors, making the story much more interesting.

What are sinusoids so important in our business?

- Almost all electrical energy generated and transmitted in the world is in the form of sinusoid voltages and currents. (DC generation – in the form of solar cells – is on the rise, but AC power is still dominant.) Rotating machines (generators) naturally produce sinusoidal voltages and currents. voltages. Also, it is easy to change voltage levels using transformers. Much of this is due to Tesla (the man, not the car company).
- We will learn in later classes (electronics, signal and systems) that all electrical waveforms can be described in terms of sinusoids. (This is Fourier analysis.) For instance, a square wave can be viewed as a combination of many sinusoids of various frequencies and amplitudes. This concept is at the heart of encoding information into electrical signals. Since we can express every waveform in terms of sinusoids, once we know how a circuit responds to a sinusoid, we can know how it will respond to any kind of waveform.

Sinusoidal sources

Independent voltage and current sources will be time-dependent of the form

$$V_S(t) = V_m \cos \omega t \qquad\qquad I_S(t) = I_m \sin \omega t$$

As we proceed, we will learn that it doesn't really matter whether a source is a sine or cosine. For a single source, the 90° phase difference between a sine or cosine is irrelevant. It is true that in a sinusoidal circuit, the phases of all other voltages and currents in the circuit are *extremely* important. But since the source is the forcing function in the circuit, all other voltages and currents will come into alignment with it, according Kirchoff's Laws. The frequency will be the same everywhere, and the amplitudes and phase differences will be with respect to whatever the source is doing. If the phase of the source is shifted by 90°, everything in the circuit will shift by 90° in response.

There is no absolute value of phase — only phase differences matter. (Phase is like energy in that respect.)

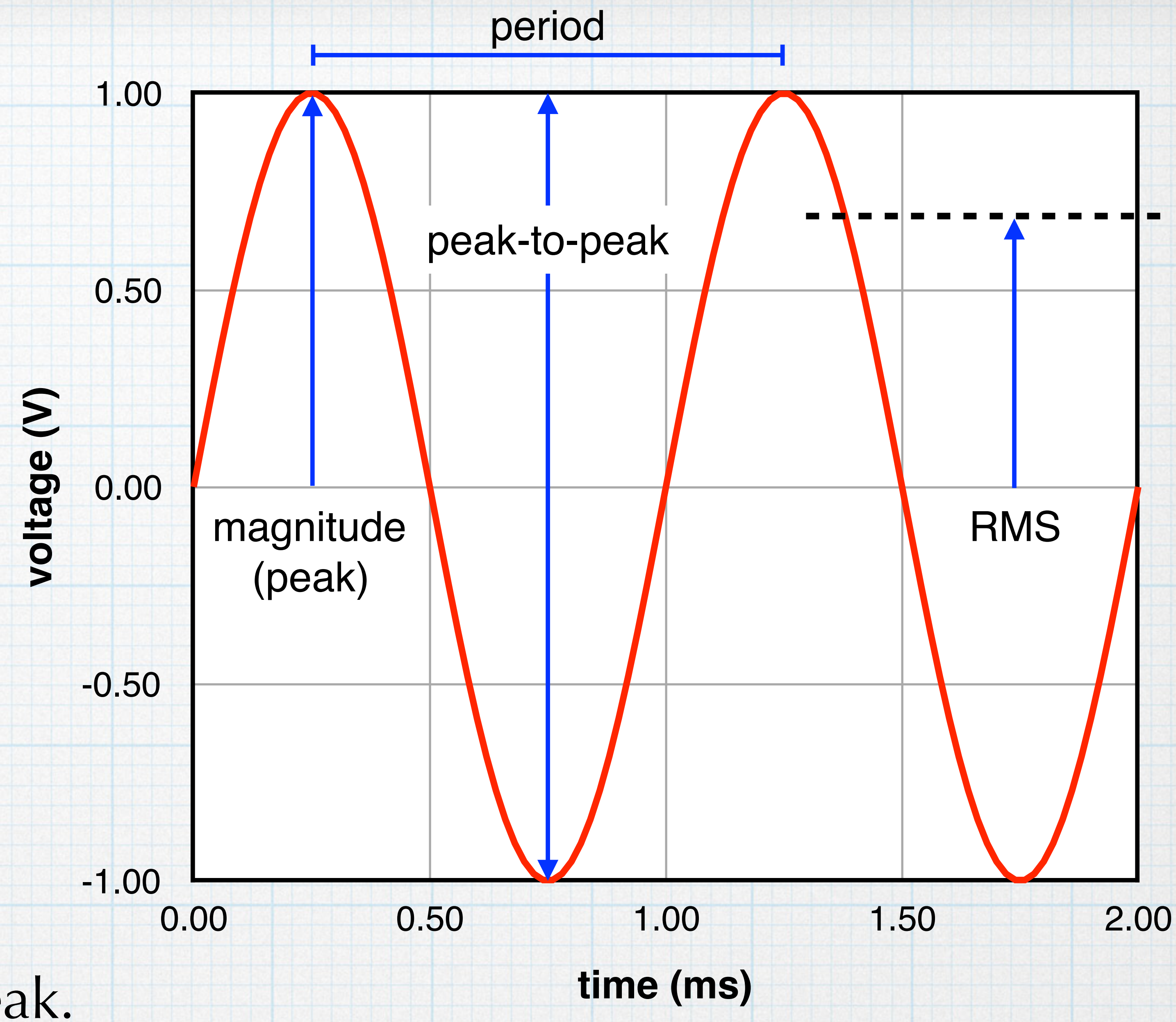
Plot of sinusoidal waveform.

$$V_m \sin\left(\frac{2\pi}{T}\right)$$

$$V_m \sin\left(2\pi f t\right)$$

$$V_m \sin \omega t$$

From the graph,
$V_m = 1$ V.



V_m – magnitude or amplitude or peak.

Also, peak-to-peak = $2 \cdot V_m$ (Used by function generator.) Here, $V_m = 2\ \mathrm{V_{pp}}$

Also, RMS = $V_m/\sqrt{2}$ (Used by multimeter.) Here, $V_m = 0.707\ \mathrm{V_{RMS}}$

T – period. Time for one complete cycle. From the graph, $T = 1$ ms.

f – frequency. Cycles in one second. $f = 1/T$. Here $f = 1/(1\text{ms}) = 1000$ Hz.

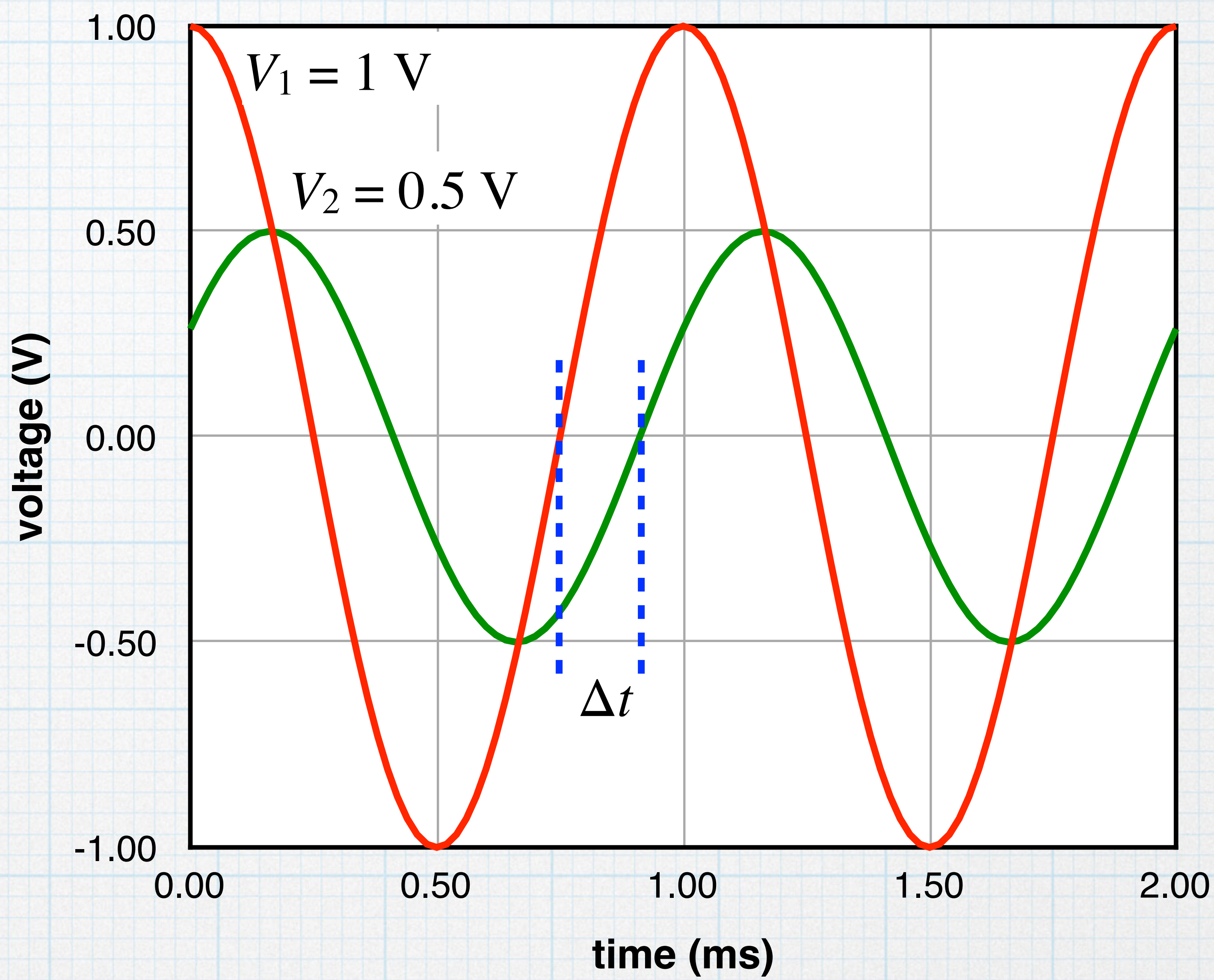
ω – angular frequency. Radians in one second. $\omega = 2\pi f$. Here $\omega = 6283$ rad/s.

Phase angle

$$v_1 = V_1 \cos \omega t$$

$$v_2 = V_2 \cos(\omega t - \theta)$$

$T = 1$ ms, $f = 1$ kHz.

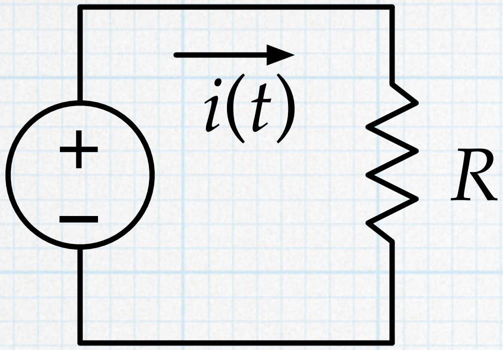


$$\theta = \omega \Delta t = \frac{2\pi}{T} \Delta t = 2\pi f \Delta t$$

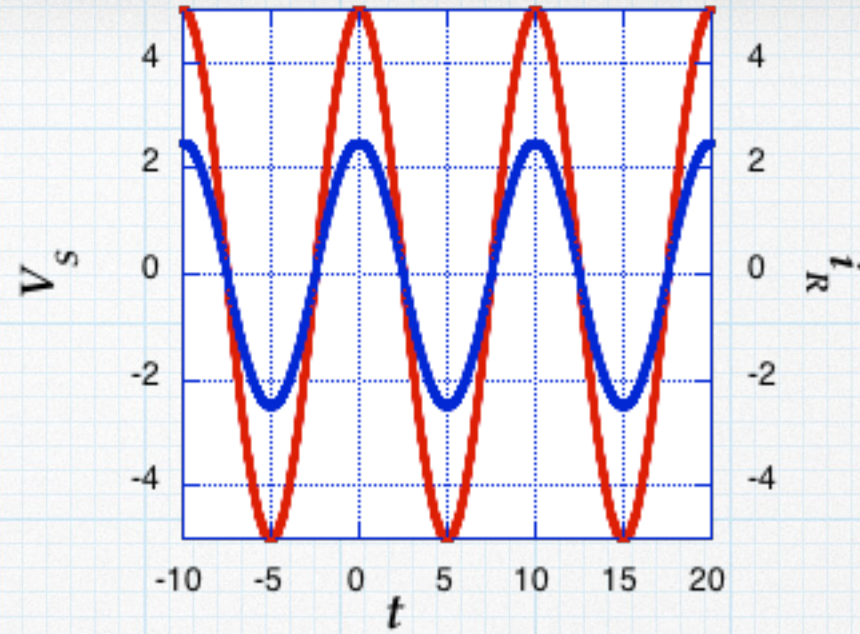
$$\theta = (360^\circ) \frac{\Delta t}{T} = (360^\circ) f \Delta t$$

$\theta = \pi/4 = 45° \rightarrow \Delta t = 0.125$ ms.

$V_S(t) = V_m \cos \omega t$

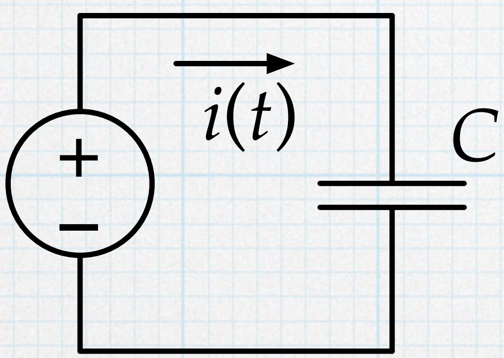


$$i\,(t) = \frac{V_m}{R}\cos \omega t$$

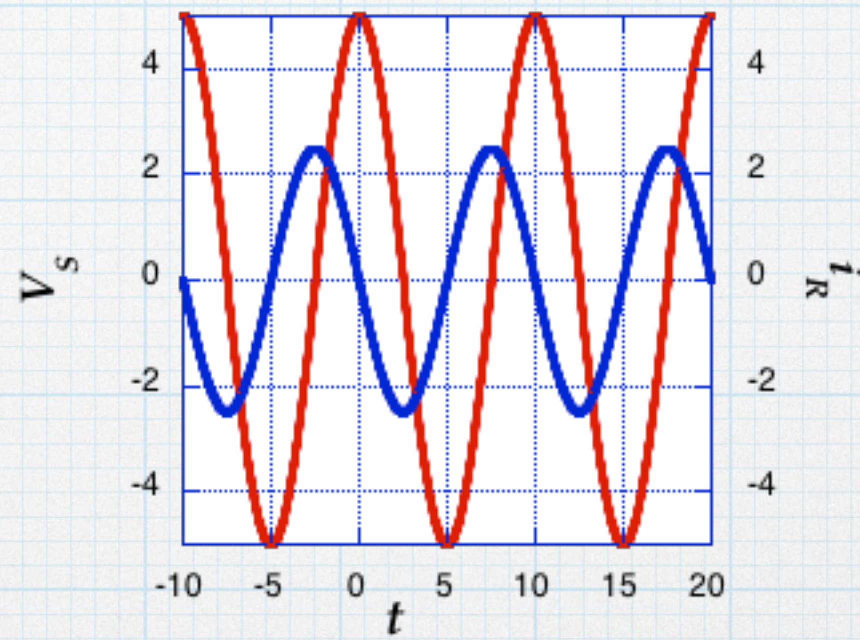


Current in phase with voltage.

$V_S(t) = V_m \cos \omega t$

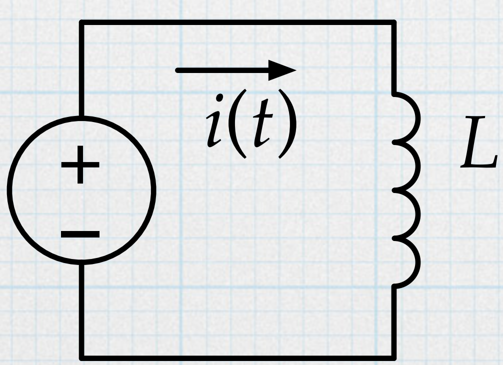


$$i\,(t) = -\omega C V_m \sin \omega t$$

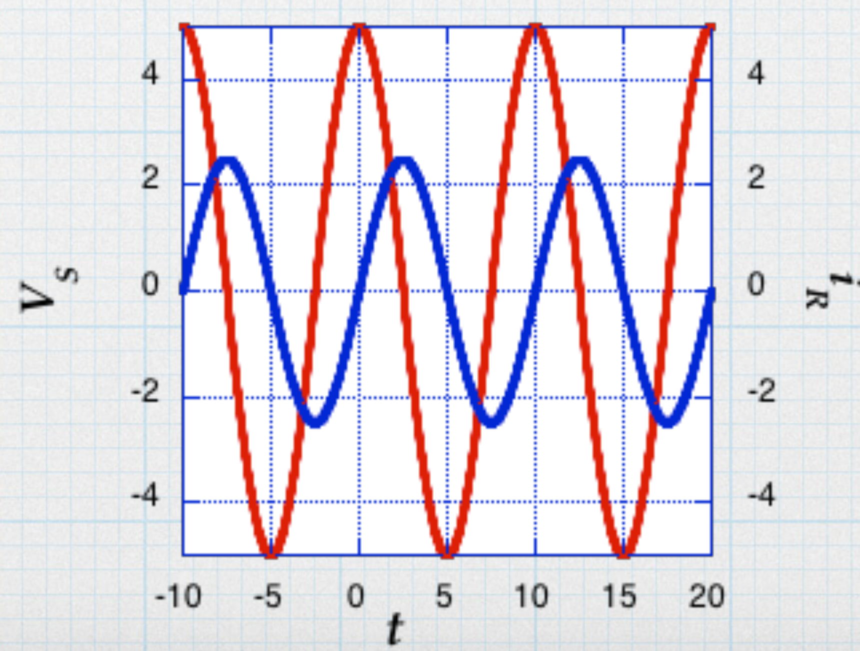


Current *leads* voltage by 90°.

$V_S(t) = V_m \cos \omega t$



$$i\,(t) = \frac{V_m}{\omega L}\sin \omega t$$

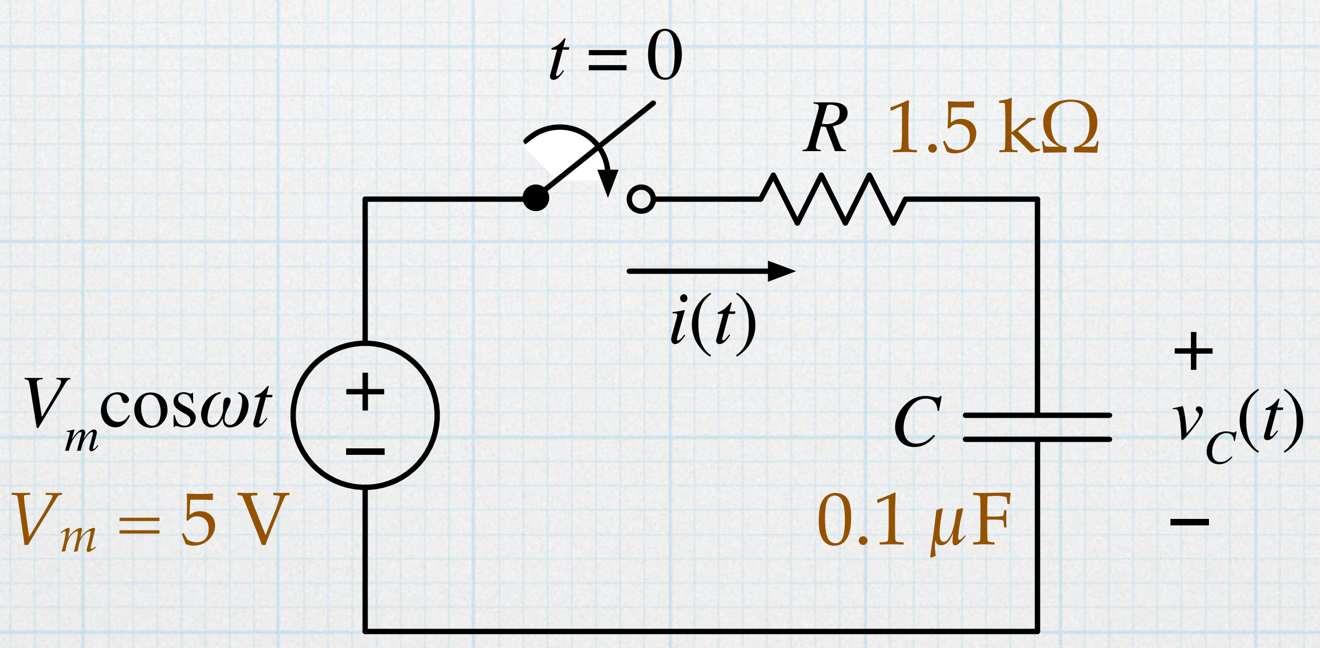


Current *lags* voltage by 90°.

Example: RC circuit

It's time to take the plunge and solve a circuit that has a mixture of components driven by a sinusoidal source. Rather than plunging, perhaps we should step gingerly and try a relatively simple circuit — a resistor and a capacitor in series, driven by a sinusoidal voltage source, as shown below.

The circuit looks to be similar to the *RC* version of the step-function transient circuits we studied earlier. But the results will be quite different due to the sinusoidal source, which will become the forcing function in the upcoming differential equation.

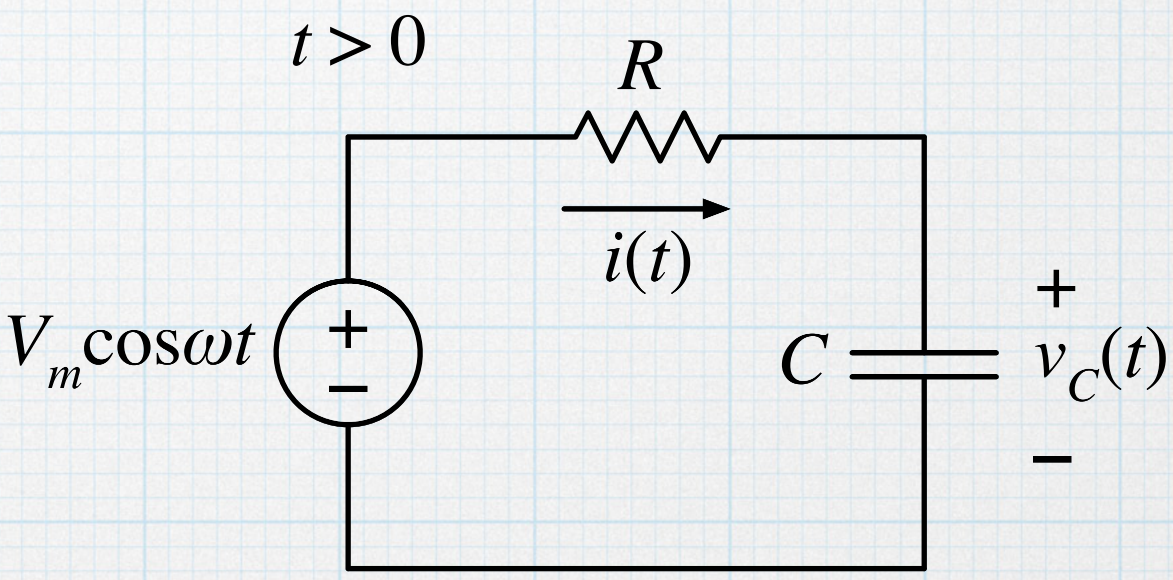


$\omega = 6660$ rad/s

($f = 1060$ Hz. $T = 0.943$ ms)

We assume the switch closes exactly at $t = 0$, when the source voltage is at its very peak. Obviously, this might be rather tricky in real life, but on paper we can assume that it will happen. Shortly, we will invoke a simplification that makes the timing of the switch irrelevant. Also, we will use the initial condition that the capacitor voltage is 0 before the switch closes, and so will still be at zero at the instant after the switch has closed. Then initial condition for the capacitor is $v_C(t = 0) = 0$.

Start with KVL:



$$i_R = i_C$$

$$\frac{V_m \cos \omega t - v_C(t)}{R} = C\frac{dv_C(t)}{dt}$$

$$\frac{dv_C(t)}{dt} + \frac{v_C(t)}{RC} = \frac{V_m(t)}{RC} \cos \omega t$$

Use the two-solution approach: $v_c(t) = v_{tr}(t) + v_{SS}(t)$.

$$\frac{dv_{tr}(t)}{dt} + \frac{v_{tr}(t)}{RC} = 0 \qquad \frac{dv_{ss}(t)}{dt} + \frac{v_{ss}(t)}{RC} = V_m \cos \omega t$$

Start with transient (homogeneous) part.

guess → $v_{tr} = D \exp(st)$

See if there is a value of s that works. Determine D later using initial conditions. Insert the guess into the homogeneous equation.

$$sDe^{st} + \frac{D}{RC} e^{st} = 0$$

$$s + \frac{1}{RC} = 0$$

$$s = -\frac{1}{RC} = -\frac{1}{\tau}$$

$$v_{tr} = D \exp\left(-\frac{t}{RC}\right)$$

→ The transient part is a decaying exponential, just as we have seen previously.

Now try to find a steady-state solution.

$$\frac{dv_{ss}(t)}{dt} + \frac{v_{ss}(t)}{RC} = V_m \cos \omega t$$

Given that the forcing function is cos(ωt), it might seem reasonable to use $v_{ss}(t) = A\cos(\omega t)$ as a trial function and then try to find a value of A that works. Plugging the trial function into the differential equation:

$$-\omega A \sin \omega t + \frac{A}{RC} \cos \omega t = \frac{V_m}{RC} \cos \omega t$$

That's not going to work. There is no value of A that will satisfy this equation for all values of t. It will not help to try sin(ωt) either — the derivative will generate a cosine term, leading again to a mixture of sine and cosine terms on the left side. And there are no trig identities that will get us out of this jam. This is a dead end.

Because the derivative in the steady-state equation introduces a term with sin(ωt), maybe we need to make our trial function a bit more inclusive and have 2 sinusoidal terms.

$$v_{ss}(t) = A \cos \omega t + B \sin \omega t$$

trial function: $v_{ss}(t) = A\cos\omega t + B\sin\omega t$

Substituting this into the steady-state equation makes things a bit more complicated, but also gives us a route to a solution.

$$-\omega A\sin\omega t + \omega B\cos\omega t + \frac{A}{RC}\cos\omega t + \frac{B}{RC}\sin\omega t = \frac{V_m}{RC}\cos\omega t$$

Gathering sine and cosine terms,

$$\left[\frac{A}{RC} + \omega B\right]\cos\omega t + \left[-\omega A + \frac{B}{RC} +\right]\sin\omega t = \frac{V_m}{RC}\cos\omega t$$

We see that things will work out if

$$\frac{A}{RC} + \omega B = \frac{V_m}{RC} \qquad \text{and} \qquad -\omega A + \frac{B}{RC} = 0$$

These are two equations in the two unknowns, A and B. We should be good at solving such problems by now.

Solving the two equations for A and B in terms of R, C, ω, and V_m:

$$A = \frac{1}{1+(\omega RC)^2} V_m \qquad B = \frac{\omega RC}{1+(\omega RC)^2} V_m$$

These expression look a bit messy, but with A and B as given, the trial solution matches the requirements for being a steady-state solution to the differential equation.

$$v_{ss}(t) = \left[\frac{V_m}{1+(\omega RC)^2}\right] \cos \omega t + \left[\frac{\omega RC V_m}{1+(\omega RC)^2}\right] \sin \omega t$$

Then, the complete solution is:

$$v_C(t) = \left[\frac{V_m}{1+(\omega RC)^2}\right] \cos \omega t + \left[\frac{\omega RC V_m}{1+(\omega RC)^2}\right] \sin \omega t + D \exp\left(-\frac{t}{RC}\right)$$

Plugging in numbers:

$$v_C(t) = (2.5\,\mathrm{V}) \cos \omega t + (2.5\,\mathrm{V}) \sin \omega t + D \exp\left(-\frac{t}{RC}\right)$$

All that is left is to determine D. Using the initial condition:

$$v_C(t = 0) = 0 = A + D$$

Then, $D = -A = -2.5$ V.

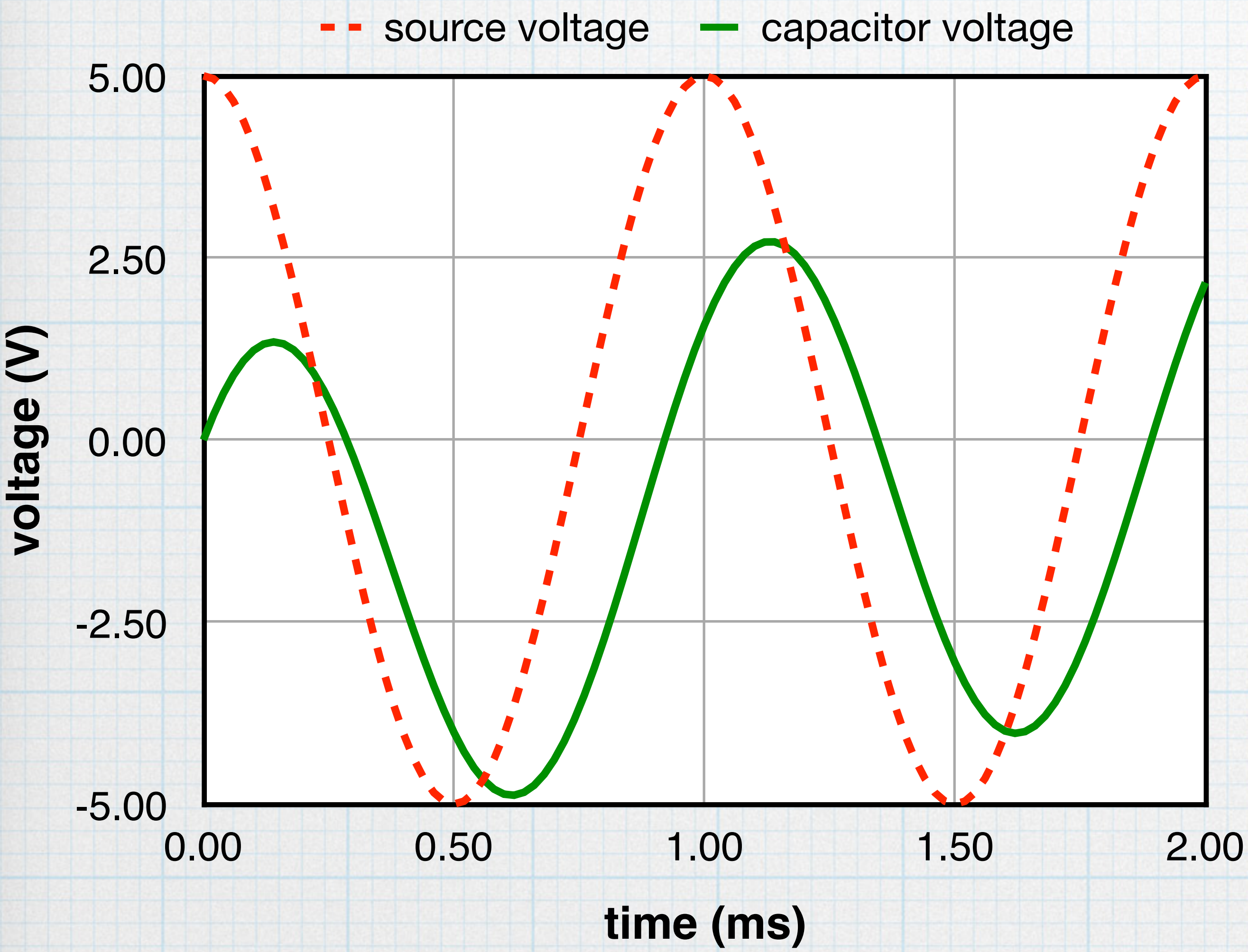
$$v_C(t) = \left(2.5\,\mathrm{V}\right)\cos\omega t + \left(2.5\,\mathrm{V}\right)\sin\omega t - \left(2.5\,\mathrm{V}\right)\exp\left(-\frac{t}{\tau}\right)$$

$\omega = 6660$ rad/s and $\tau = 1.5$ ms.

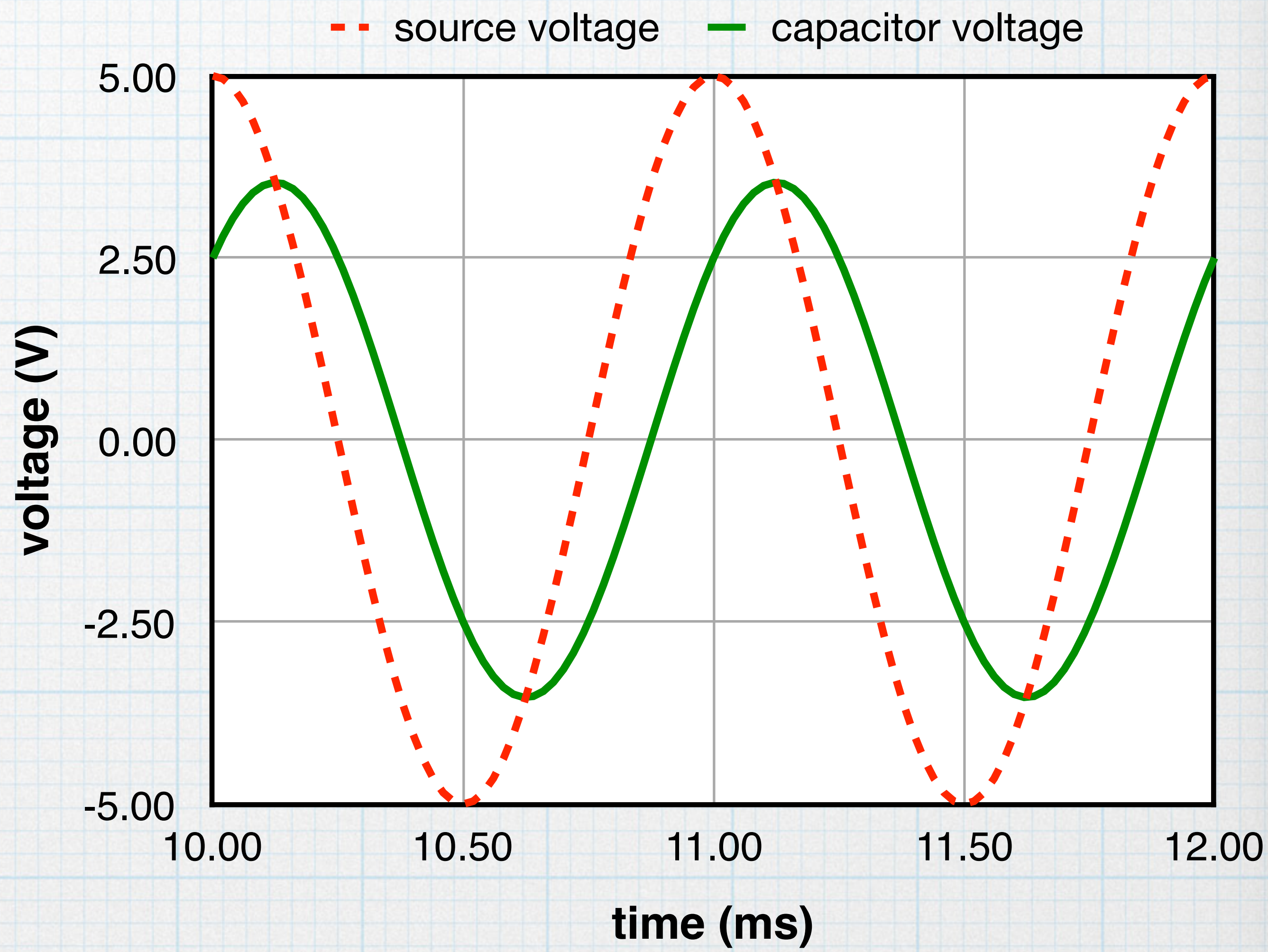
That's the whole shebang.

As we saw with RC step-change transients, after 5 time constants (5 ms in this case), the exponential term will decay away to essentially nothing. After the transient has faded, the capacitor voltage will oscillate in a sinusoidal fashion forever, driven by the source voltage.

Below are two plots of the source and capacitor voltages. The left one starts at zero and shows the transient at the startup. The right plot shows a couple of periods well beyond the 5 time constants of the transient, where we can see the steady-state situation.



$0 < t < 2$ ms (transient)



10 ms $< t <$ 12 ms (steady state)

Examining the steady-state plots, we note something interesting. We know that the capacitor voltage consists of a sum of two sinusoids. Yet, the capacitor trace looks like a single sinusoid at the same frequency as the source voltage (which is to be expected), but reduced in amplitude and shifted in phase.

Indeed, that is exactly the case. Two pure sinusoids oscillating at the same frequency will always add to produce a single sinusoid with a different amplitude and possibly a phase shift.

There is a trig identity for this situation — it's a bit obscure, but useful in this case. Given the sum of a cosine and sine, the

$$A \cos x + B \sin x = M \cos (x - \delta x)$$

where $M = \sqrt{A^2 + B^2}$ and $\delta x = \arctan\left(\frac{B}{A}\right)$

(These relationships look vaguely familiar.)

Applying this identity to the capacitor voltage:

$$M = \sqrt{A^2 + B^2}$$

$$= \sqrt{\frac{V_m^2}{\left[1 + (\omega RC)^2\right]^2} + \frac{(\omega RC)^2 V_m^2}{\left[1 + (\omega RC)^2\right]^2}}$$

Yuck!

$$= \frac{V_m}{\sqrt{1 + (\omega RC)^2}}$$

Well, that's not so bad.

$$= 3.54 \text{ V}$$

$$\theta = \arctan\left(\frac{B}{A}\right)$$

$$= \arctan(\omega RC)$$

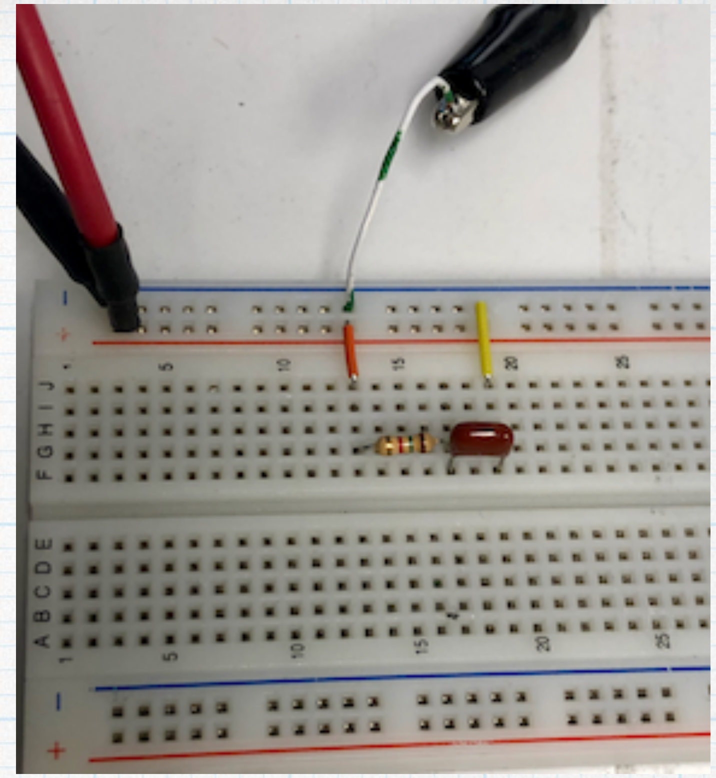
$$= 45^\circ$$

Then, the capacitor voltage can be written with one sinusoidal term:

$$v_C(t) = \left(3.54 \text{ V}\right) \cos\left(\omega t - 45^\circ\right) - \left[2.5 \text{ V}\right] e^{-\frac{t}{\tau}}$$

Measurement of the *RC* circuit in the lab. One 1.5-kΩ resistor and one 0.1-μF cap.

Yellow is the source, green is the capacitor voltage. Measured values are in the lower corner of the screen.



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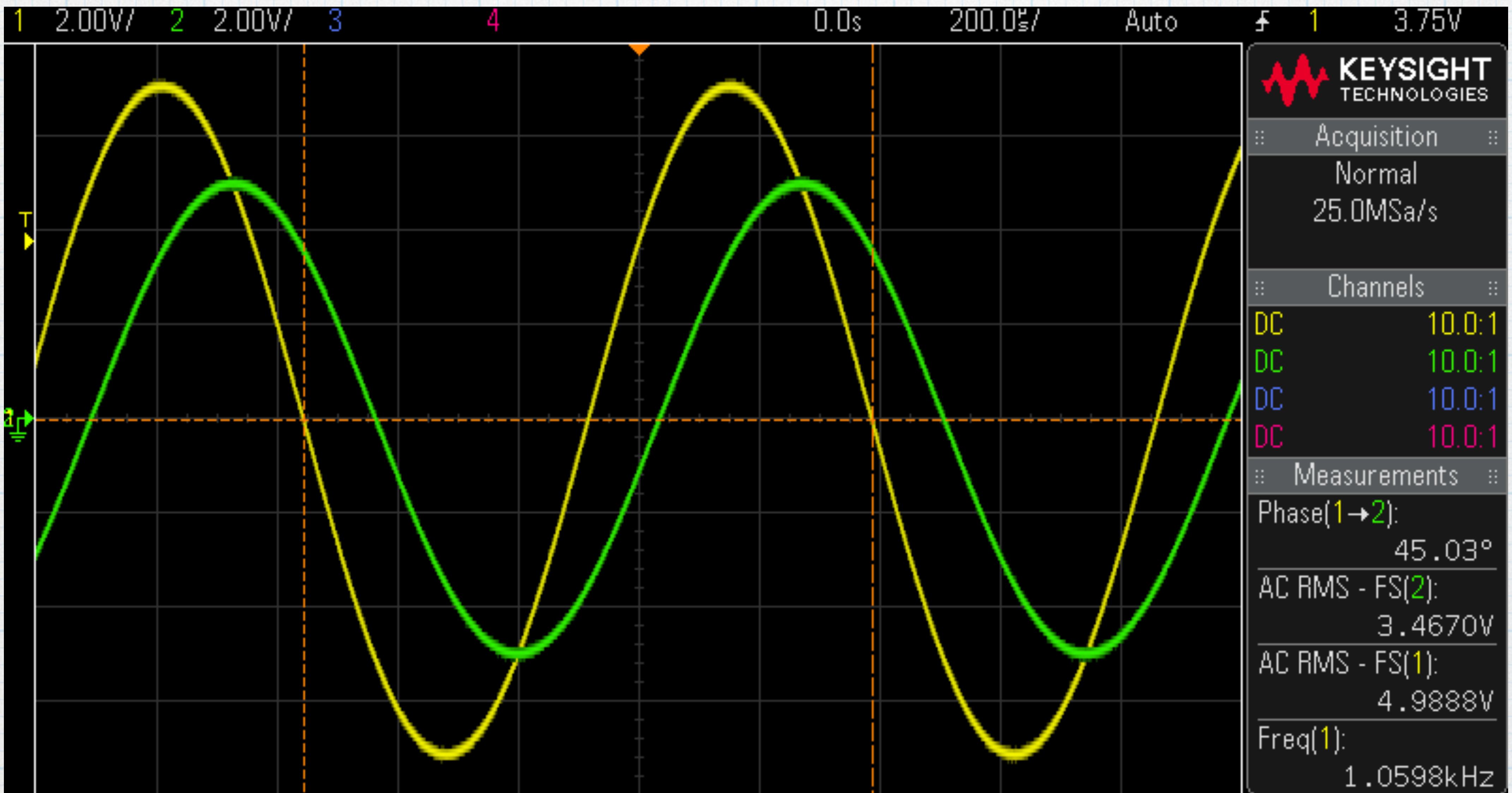
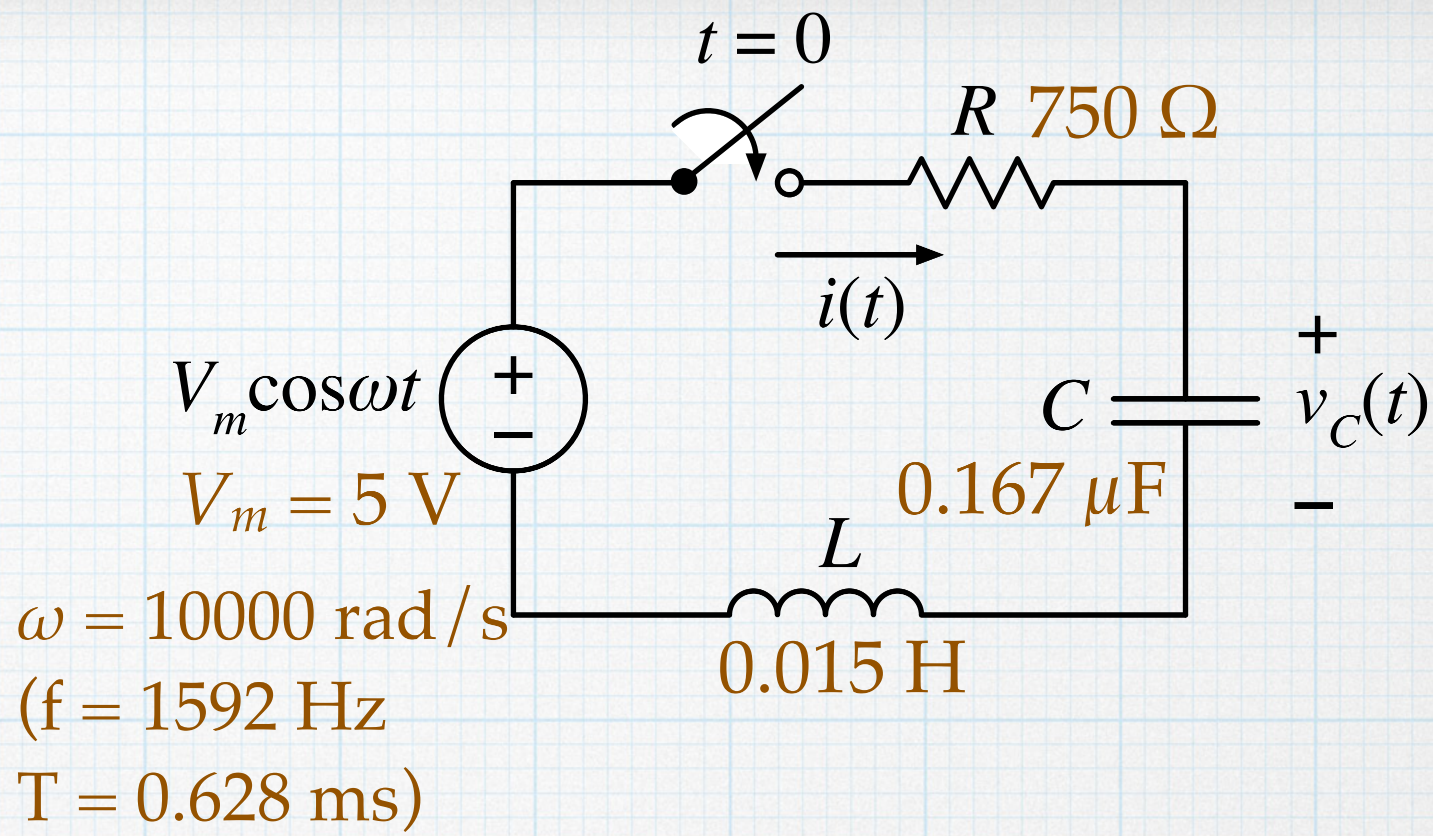


A second example

Consider an *RLC* circuit with a sinusoidal source. The switch closes at $t = 0$ (when the source is at its peak). For $t < 0$, the capacitor voltage and the current are both zero.



The *RLC* step-function transient problem was much more complicated than the first-order versions. Is a second-order sinusoidal problem significantly more complicated than a first-order version? Let's find out.

Start with KVL:

$$V_S(t) = v_R + v_C + v_L$$

$$= iR + v_C + L\frac{di}{dt}$$

Recall, $i = C\frac{dv_C}{dt}$. Substitute in.

$$= RC\frac{dv_C}{dt} + v_C + LC\frac{d^2v_C}{dt^2}$$

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{V_m}{LC}\cos\omega t$$

$$\frac{d^2 v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_m}{LC}\cos\omega t$$

Once again, we look for transient and steady-state solutions.

$$v_C(t) = v_{tr}(t) + v_{ss}(t)$$

$$\frac{d^2 v_{tr}(t)}{dt^2} + \frac{R}{L}\frac{dv_{tr}(t)}{dt} + \frac{1}{LC}v_{tr}(t) = 0 \qquad \text{(homogeneous)}$$

$$\frac{d^2 v_{ss}(t)}{dt^2} + \frac{R}{L}\frac{dv_{ss}(t)}{dt} + \frac{1}{LC}v_{ss}(t) = \frac{V_m}{LC}\cos\omega t \qquad \text{(particular)}$$

The homogeneous equation is exactly the same as we found in the earlier *RLC* step-function problem, and the solution here will be the same — over-damped or under-damped transients, depending on the amount of resistance in the circuit. In either case, the transient function will fade away after a few time constants, leaving only the steady-state response.

Let's finish the transient analysis first. From our earlier work with step-functions, we know the form for the transient function and the expressions for the two roots:

$$v_{tr}(t) = D\exp\left(s_1 t\right) + F\exp\left(s_2 t\right)$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Inserting the component values from the circuit, we find that the transient response is underdamped (both roots are real and negative), with

$s_1 = -10000\ \text{s}^{-1}$ ($\tau_1 = 0.1\ \text{ms} = 100\ \mu\text{s}$) and

$s_2 = -40000\ \text{s}^{-1}$ ($\tau_2 = 0.025\ \text{ms} = 25\ \mu\text{s}$)

The transient response will have decayed away almost completely after 0.5 ms, which less than one period of the sinusoid.

Now turn to the steady-state half of the problem.

$$\frac{d^2 v_{ss}}{dt^2} + \frac{R}{L}\frac{dv_{ss}}{dt} + \frac{1}{LC} v_{ss} = \frac{V_m}{LC}\cos\omega t$$

Again, we expect the steady-state function to follow the general from of the forcing function, but as we saw in the previous example, the derivatives on the left-hand side will produce terms that go as the $\sin\omega t$. So, as was necessary last time, it seems reasonable to use

$$v_{ss}(t) = A\cos\omega t + B\sin\omega t$$

as a trial function. Inserting the trial function into the differential equation

$$-\omega^2 \left(A\cos\omega t + B\sin\omega t\right) + \frac{\omega R}{L}\left(-A\sin\omega t + B\cos\omega t\right) + \frac{1}{LC}\left(A\cos\omega t + B\sin\omega t\right) = \frac{V_m}{LC}\cos\omega t$$

and gathering together the sine and cosine terms:

$$\left(-\omega^2 A + \frac{\omega R}{L}B + \frac{1}{LC}A\right)\cos\omega t + \left(-\omega^2 B - \frac{\omega R}{L}A + \frac{1}{LC}B\right)\sin\omega t = \frac{V_m}{LC}\cos\omega t$$

$$\left(-\omega^2 A + \frac{\omega R}{L} B + \frac{1}{LC} A\right) \cos \omega t + \left(-\omega^2 B - \frac{\omega R}{L} A + \frac{1}{LC} B\right) \sin \omega t = \frac{V_m}{LC} \cos \omega t$$

The proposed trial function will work if the coefficients A and B satisfy

$$-\omega^2 A + \frac{\omega R}{L} B + \frac{1}{LC} A = \frac{V_m}{LC}$$

$$-\omega^2 B - \frac{\omega R}{L} A + \frac{1}{LC} B = 0$$

Two equations, two unknowns. Solving:

$$A = \frac{1 - \omega^2 LC}{\left(1 - \omega^2 LC\right)^2 + (\omega RC)^2} V_m$$

$$B = \frac{\omega RC}{\left(1 - \omega^2 LC\right)^2 + (\omega RC)^2} V_m$$

Plugging in the specific values from the circuit: $A = 1.76$ V and $B = 2.94$ V.

The complete solution is:

$s_1 = -10000\ \text{s}^{-1}$

$s_2 = -40000\ \text{s}^{-1}$

$A = 1.76\ \text{V}$

$B = 2.94\ \text{V}$

$$v_C(t) = D\exp\left(s_1 t\right) + F\exp\left(s_2 t\right) + A\cos\omega t + B\sin\omega t$$

We still need to find the coefficients D and F for the transient part. We use the initial conditions to get those: $v_C(t = 0) = 0$ and $i(t = 0) = 0$. The voltage condition is straight-forward. The current condition implies that the derivative of the capacitor voltage must be 0 just after the switch: $i/C = dv_c/dt = 0$. Applying these two conditions:

$$v_C(0) = D + F + A = 0$$

$$\left.\frac{dv_C}{dt}\right|_{t=0} = s_1 D + s_2 F + \omega B = 0$$

Solving gives:

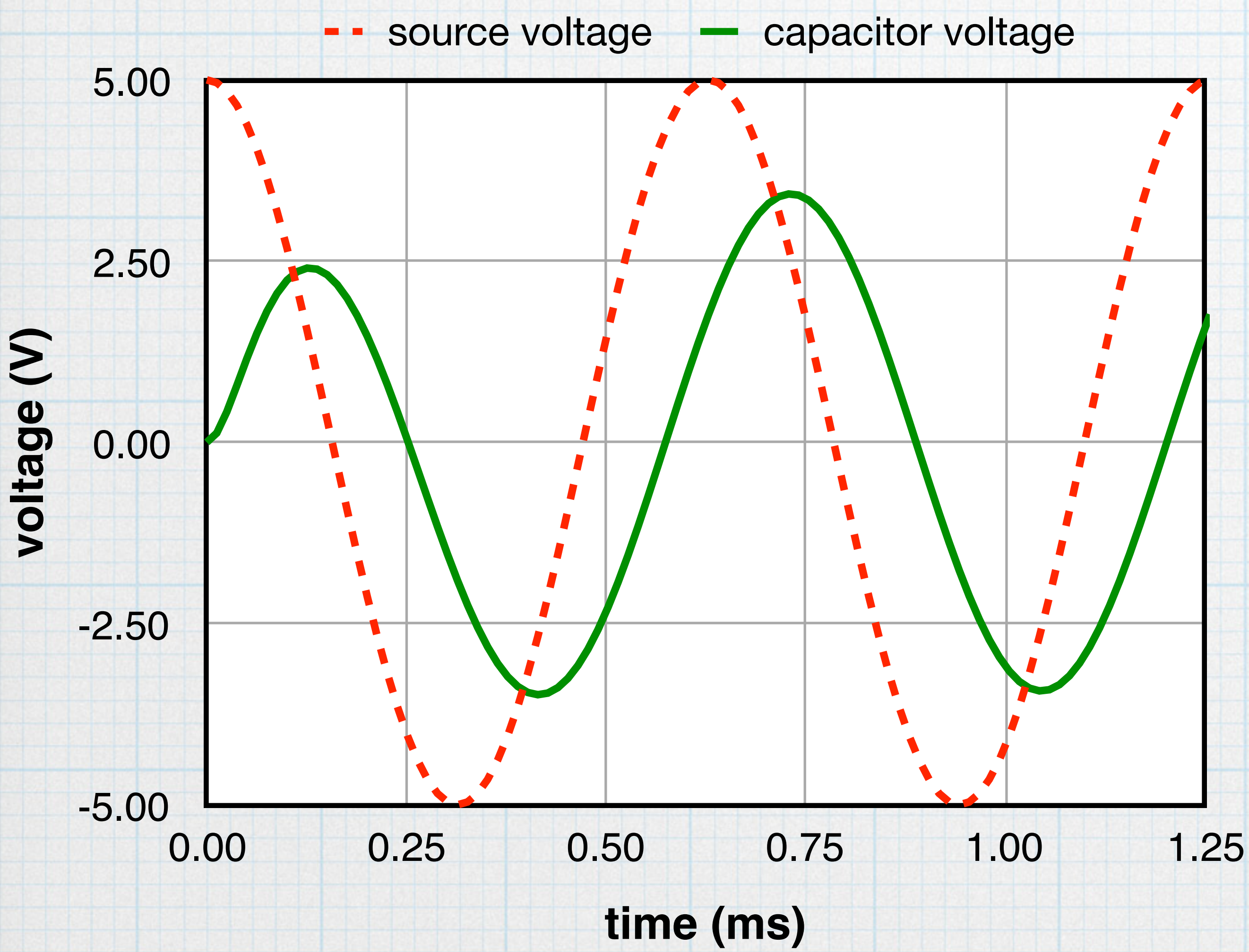
$$D = \frac{s_2 A - \omega B}{s_1 - s_2} = -\ 3.33\ \text{V}$$

$$F = \frac{s_1 A - \omega B}{s_2 - s_1} = 1.57\ \text{V}$$

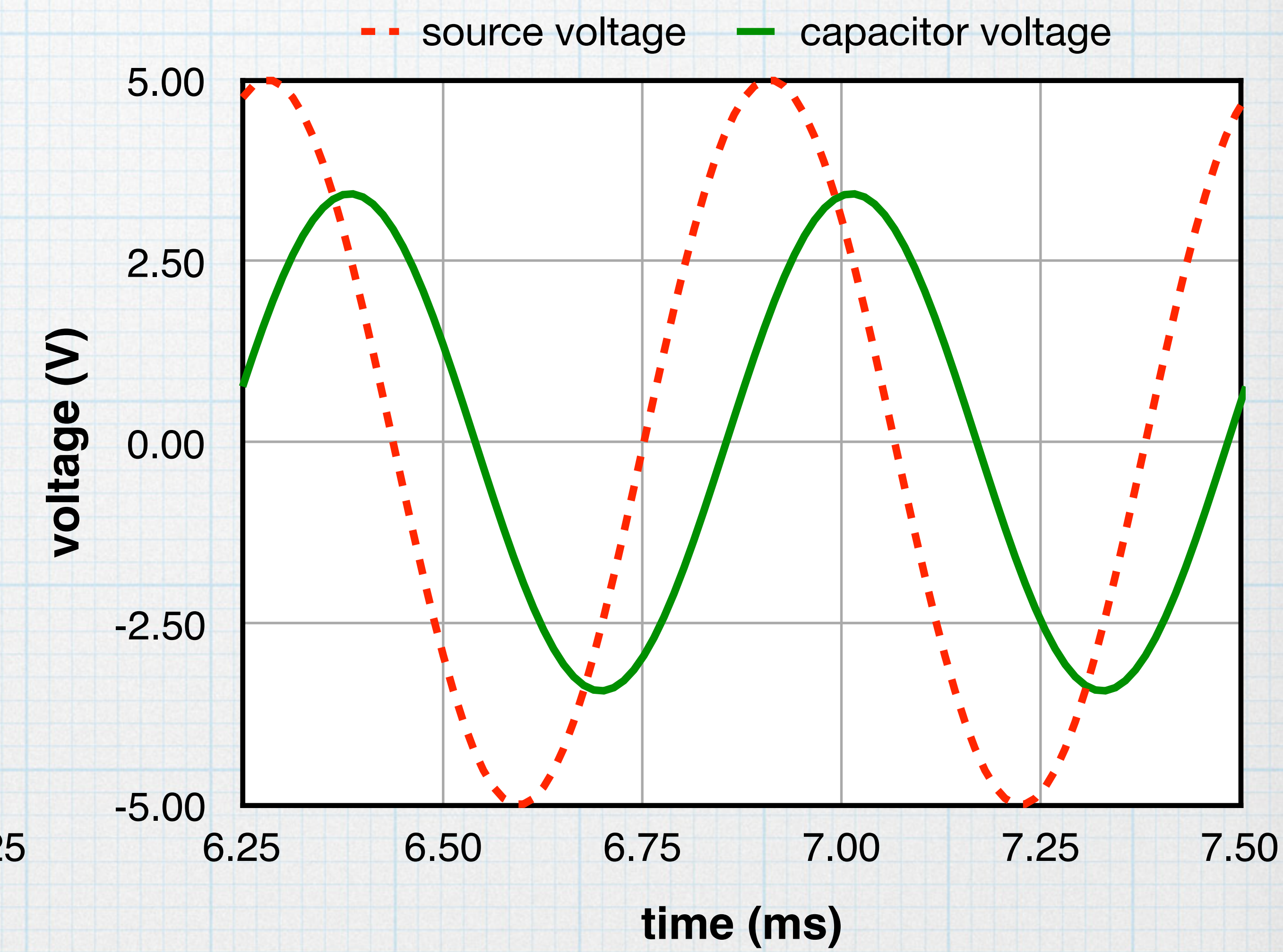
$$v_C(t) = (-3.33\ \text{V})\exp\left(s_1 t\right) + \left(1.57\ \text{V}\right)\exp\left(s_2 t\right) + \left(1.76\ \text{V}\right)\cos\omega t + (2.94\ \text{V})\sin\omega t$$

$$v_C(t) = (-3.33\text{ V})\exp\left(s_1 t\right) + \left(1.57\text{ V}\right)\exp\left(s_2 t\right) + \left(1.76\text{ V}\right)\cos\omega t + (2.94\text{ V})\sin\omega t$$

Below are two plots of the source and capacitor voltages. The left one starts at zero and shows the transient at the startup. The right plot shows a couple of periods that are well beyond the 5 time constants of the transient, where we can see the steady-state situation.



$0 < t < 1.25$ ms (transient)



6.25 ms $< t <$ 7.5 ms (steady state)

In looking at the steady-state plots we see that the capacitor voltage is a sinusoid at the same frequency as the source, but with reduced amplitude and shifted in phase. Interestingly, the general behavior of the steady-state function is just like that of the simpler *RC* circuit of example 1. The details (amplitude and phase shift) are different because they are different circuits, but the qualitative features are similar.

We can use the "sum of two sinusoids" identity introduced earlier to find the magnitude and phase shift of the capacitor voltage,

$$M = \sqrt{A^2 + B^2}$$

$$= \sqrt{\frac{\left(1-\omega^2 LC\right)^2 V_m^2}{\left[\left(1-\omega^2 LC\right)^2 + (\omega RC)^2\right]^2} + \frac{(\omega RC)^2 V_m^2}{\left[\left(1-\omega^2 LC\right)^2 + (\omega RC)^2\right]^2}} \quad \text{(Yikes!!)}$$

$$= \frac{V_m}{\sqrt{\left(1-\omega^2 LC\right)^2 + (\omega RC)^2}}$$

$$= 3.43 \text{ V}$$

$$\theta = \arctan\left(\frac{B}{A}\right)$$

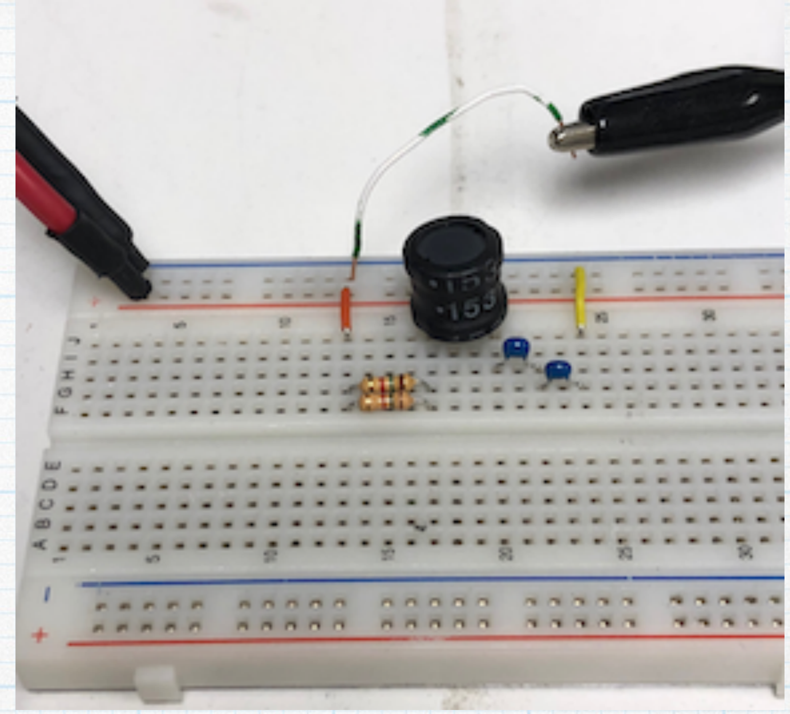
$$\theta = \arctan\left(\frac{\omega RC}{1-\omega^2 LC}\right)$$

$$= 59°$$

$$v_{tr} = (3.43 \text{ V}) \cos\left(\omega t - 59°\right)$$

Measurement of the *RLC* circuit in the lab. Two 1.5-kΩ resistors in parallel = 750 Ω and two 0.33-μF caps in series = 0.167 μF. The inductor is 15 mH.

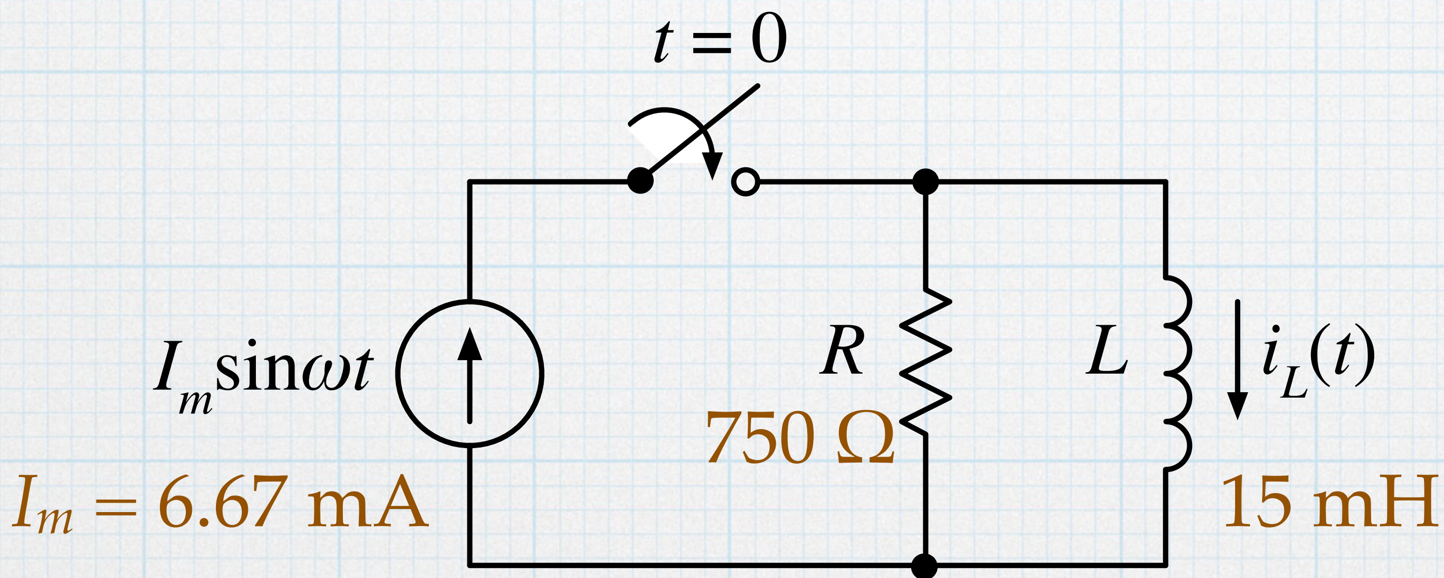
Yellow is the source, green is the capacitor voltage. Measured values are in the lower corner of the screen.



(Probes removed.)

Example 3: RL circuit

In the previous two sinusoid examples, we slashed through a *lot* of math, but in the end the steady-state responses of the capacitor voltages were quite similar — sinusoids with slightly reduced amplitudes and a phase that is shifted somewhat from the source. Is the similarity a fluke? Does it have something to do with capacitors? Or sinusoidal voltage sources? Is it the cosine function? To check, let's switch it up and try an *RL* combination driven by a current source that has $\sin\omega t$ dependence.

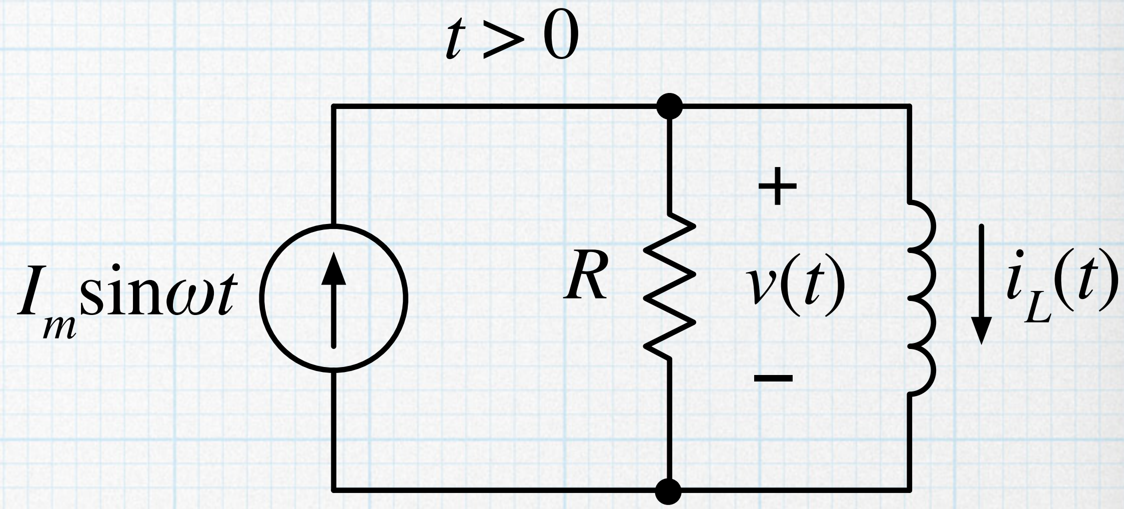


$\omega = 50000\text{ rad/s}$

$(f = 7958\text{ Hz. } T = 0.126\text{ ms})$

In the circuit, the switch closes at $t = 0$. The inductor current is zero for $t < 0$, and so the initial condition is $i_L(t = 0) = 0$.

We will not go through the details here. However, the general form of the results are nearly identical to the *RC* and *RLC* circuits — the inductor current consists of a transient plus a sinusoid that has a reduced amplitude and is phase shifted from the source sinusoid. The final results are given below.



$$i_L(t) = A\cos\omega t + B\sin\omega t + D\exp\left(-\frac{t}{\tau}\right) = M\cos(\omega t - \theta) + D\exp\left(-\frac{t}{\tau}\right)$$

$$A = -\frac{\left(\frac{\omega L}{R}\right)I_m}{1+\left(\frac{\omega L}{R}\right)^2} = -3.33\,\text{mA}$$

$$M = \frac{I_m}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} = 4.71\,\text{mA}$$

$$B = \frac{I_m}{1+\left(\frac{\omega L}{R}\right)^2} = 3.33\,\text{mA}$$

$$\theta = \arctan\left(\frac{1}{-\frac{\omega L}{R}}\right) = 135^\circ$$

$$D = -A = 3.33\,\text{mA} \qquad \tau = \frac{L}{R} = 20\,\mu\text{s}$$

(Mind the quadrant!!)

$$i_L(t) = -(3.33\,\mathrm{mA})\cos\omega t + (3.33\,\mathrm{mA})\sin\omega t + (3.33\,\mathrm{mA})\exp\left(-\frac{t}{20\,\mu\mathrm{s}}\right)$$

$$i_L(t) = (4.71\,\mathrm{mA})\cos\left(\omega t - 135^\circ\right) + (3.33\,\mathrm{mA})\exp\left(-\frac{t}{20\,\mu\mathrm{s}}\right)$$