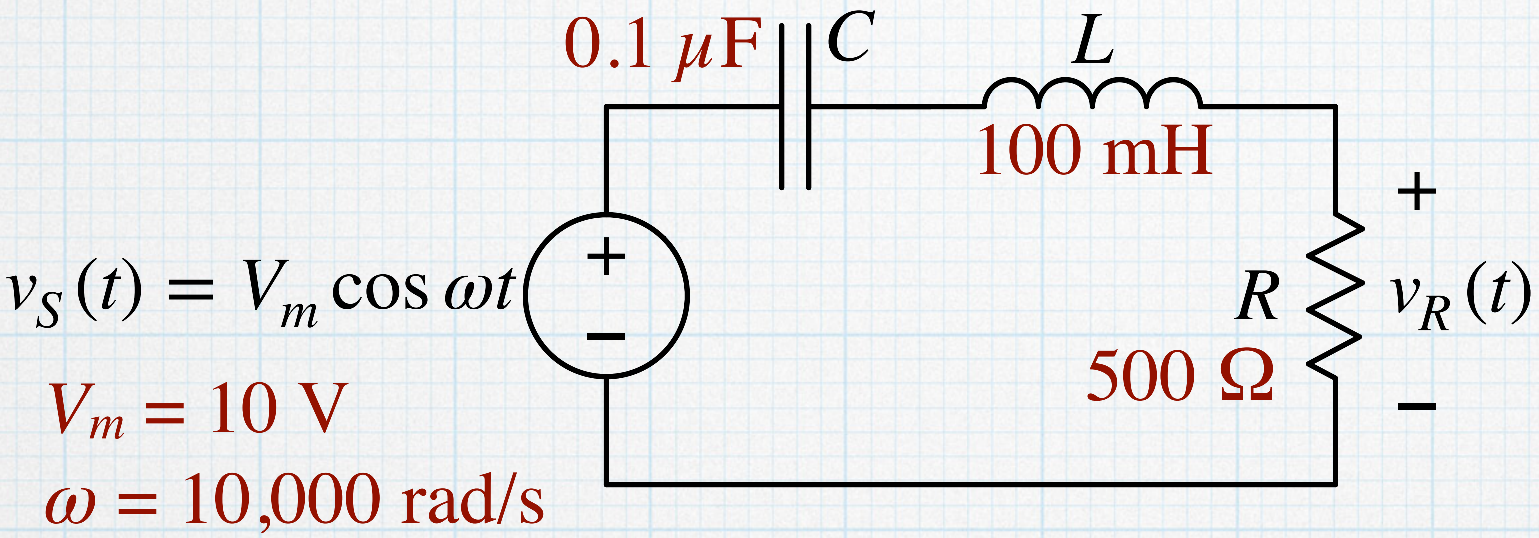
AC resonance

Consider the circuit shown below. We would like find the AC voltage across the resistor — magnitude and phase. We have done this before.



The complex voltage on the resistor is easily found using a voltage divider with the three impedances: Z_C, Z_L, and Z_R.

$$\tilde{V}_R = \frac{Z_R}{Z_C + Z_L + Z_R}\tilde{V}_S = \frac{R}{\frac{1}{j\omega C} + j\omega L + R}V_m$$

Re-arranging:

$$\tilde{V}_R = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

$$\tilde{V}_R = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \tilde{V}_S$$

That seems simple enough. Now plug in the numbers:

$$\tilde{V}_R = \frac{500\,\Omega}{500\,\Omega + j\,(1000\,\Omega - 1000\,\Omega)}\,(10\,\text{V}) = 10\,\text{V}$$

Remarkable! The resistor voltage has the same magnitude and the source and is exactly in phase. It is as if the inductor and capacitor have disappeared from the circuit.

The source of this unusual result is readily apparent:

The impedance for the inductor happens to have exactly the same magnitude as the impedance for the capacitor. Since the two have opposite signs, they have canceled each other out in the voltage divider equation, leaving only the resistor. We call this phenomenon *series resonance*.

It is easy to see the condition for series resonance. Given a capacitor and inductor in series, the combined impedance is

$$Z_{LC} = j\omega L + \frac{1}{j\omega C} = j\omega L \left(1 - \frac{1}{\omega^2 LC} \right)$$

Obviously, the combination will be equal to zero when $\omega^2 LC = 1$.

So given an AC problem with a specific operating frequency, we can always come up with an *LC* combination that creates series resonance.

Conversely, given an *LC* series combination, there is always a specific frequency at which resonance will occur:

$$\omega_o = \frac{1}{\sqrt{LC}}$$ — the resonant frequency

Any circuit with an inductor/capacitor can experience resonance, and it can cause strange effects that may surprise us if we are not on the lookout for it. On the other hand, resonance can be extremely useful if we know how to exploit it. We will see examples of using resonance when maximizing AC power transfer, building filter circuits, or making electronic circuits that create sinusoidal voltages (oscillators).

We might naively jump to the conclusion that, since the series *LC* combination has no voltage across it, the individual voltages across the inductor and capacitor are also both zero. This is most definitely incorrect! In fact, the actual values might be quite surprising.

To find the inductor and capacitor voltages, we could use the voltage divider equation again, but in this case, it is probably easier to find the current in the series circuit and then find the voltages from that.

$$\tilde{I}_S = \frac{\tilde{V}_R}{R} = \frac{10\,\text{V}}{500\,\Omega} = 20\,\text{mA}$$

(i.e. magnitude of 20 mA with and angle of 0°)

Then

$$\tilde{V}_L = Z_L \cdot \tilde{I}_S = \left(j\omega L\right)\tilde{I}_S = \left(j1000\,\Omega\right)(20\,\text{mA}) = j20\,\text{V}$$

$$\tilde{V}_C = Z_C \cdot \tilde{I}_S = \left(\frac{1}{j\omega C}\right)\tilde{I}_S = \left(-j1000\,\Omega\right)(20\,\text{mA}) = -j20\,\text{V}$$

The magnitudes of the inductor and capacitor voltages are bigger than the magnitude of the source voltage! Can this be correct? It seems that there is too much energy in the circuit — is there some sort of amplification?

There is no amplification and all of the energy/power comes from the source. The apparent conundrum results from our use of complex AC analysis to analyze the circuit. One of the basic assumptions of AC analysis was to ignore the transient phase of the circuit and look only at the steady-state behavior. By ignoring the transient, we don't see the gradual build-up of energy in the reactive components — we see only the final result in which just the right amount of energy is "stored" in the circuit. Some energy is in the inductor and some in the capacitor and they constantly trade energy back and forth, maintaining an exact balance.

An analogy is a swing at a children's playground. The rider can "pump up" the swing by shifting their weight back and forth at a rate that matches the natural oscillation frequency of the pendulum. The rider adds energy during each "pump", building up the amplitude of the swing. At steady state, the accumulated energy in the system (gravitational potential and kinetic) is traded back and forth as the pendulum swings. The rider continues to add a bit of energy during each oscillation to compensate for any losses.

There are many similar situations in electrical engineering systems — electronic oscillator circuits and lasers are two common examples.

In order to see the build-up of the energy in the resonant system, we must step back from complex AC analysis for a moment and return to the full solution as a function of time, including the transient.

We will find the complete solution for the current in the circuit, and from that we can find any of the component voltages. Recall that we need two functions for the complete solution, the steady-state and the transient. We can find the steady-state current using the steady-state voltage:

$$\tilde{i} = \frac{\tilde{V}_R}{R} = \frac{V_m}{R} = \frac{10\,\mathrm{V}}{500\,\Omega} = 20\,\mathrm{mA} \rightarrow i_{ss}(t) = I_m \cos \omega t$$

where $I_m = 20$ mA. Finding the transient function requires the usual tedious business of finding the characteristic roots of the differential equation and matching initial conditions. In this case, we assume that both the current and its derivative are zero at $t = 0$.

$$i(0) = 0 \text{ and } \left.\frac{di}{dt}\right|_{t=0} = 0.$$

We won't include the entire procedure here, but it is very similar to what we did earlier when we examined transient response in second-order circuits. (Work through the details as an exercise.)

It turns out the circuit is under-damped and the transient solution is:

$$i_{tr}(t) = I_m e^{-\sigma t}\left(-\cos\omega_d t - \frac{\sigma}{\omega}\sin\omega_d t\right)$$

where $\sigma = \frac{R}{2L}$ is the damping factor and $\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ is the damped oscillation frequency. The complete solution is the sum of the transient and steady-state functions:

$$i_{tr} = I_m e^{-\sigma t}\left(-\cos\omega_d t - \frac{\sigma}{\omega}\sin\omega_d t\right) + I_m\cos\omega t$$

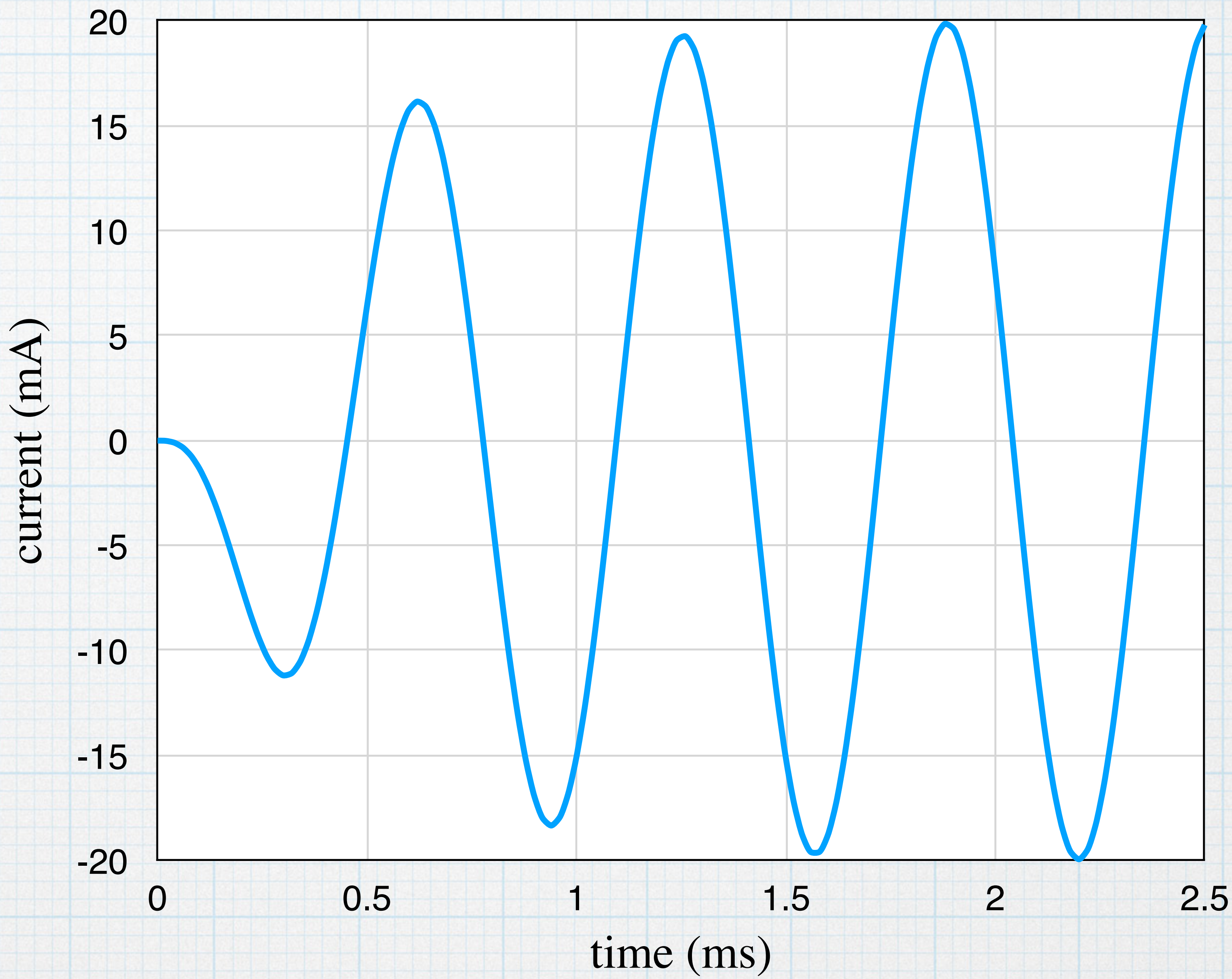
(Reminder: ω is the source frequency which is set to the resonance frequency and ω_d is the damped frequency of the transient — they are not the same!)

From the components, we can calculate the parameters in the equation:

$\sigma = 2500\,\mathrm{s}^{-1}$ (corresponding to a decay time constant of 0.4 ms) and $\omega_d = 9682\,\mathrm{rad/s}$ — close to, but not equal to — the resonant frequency.

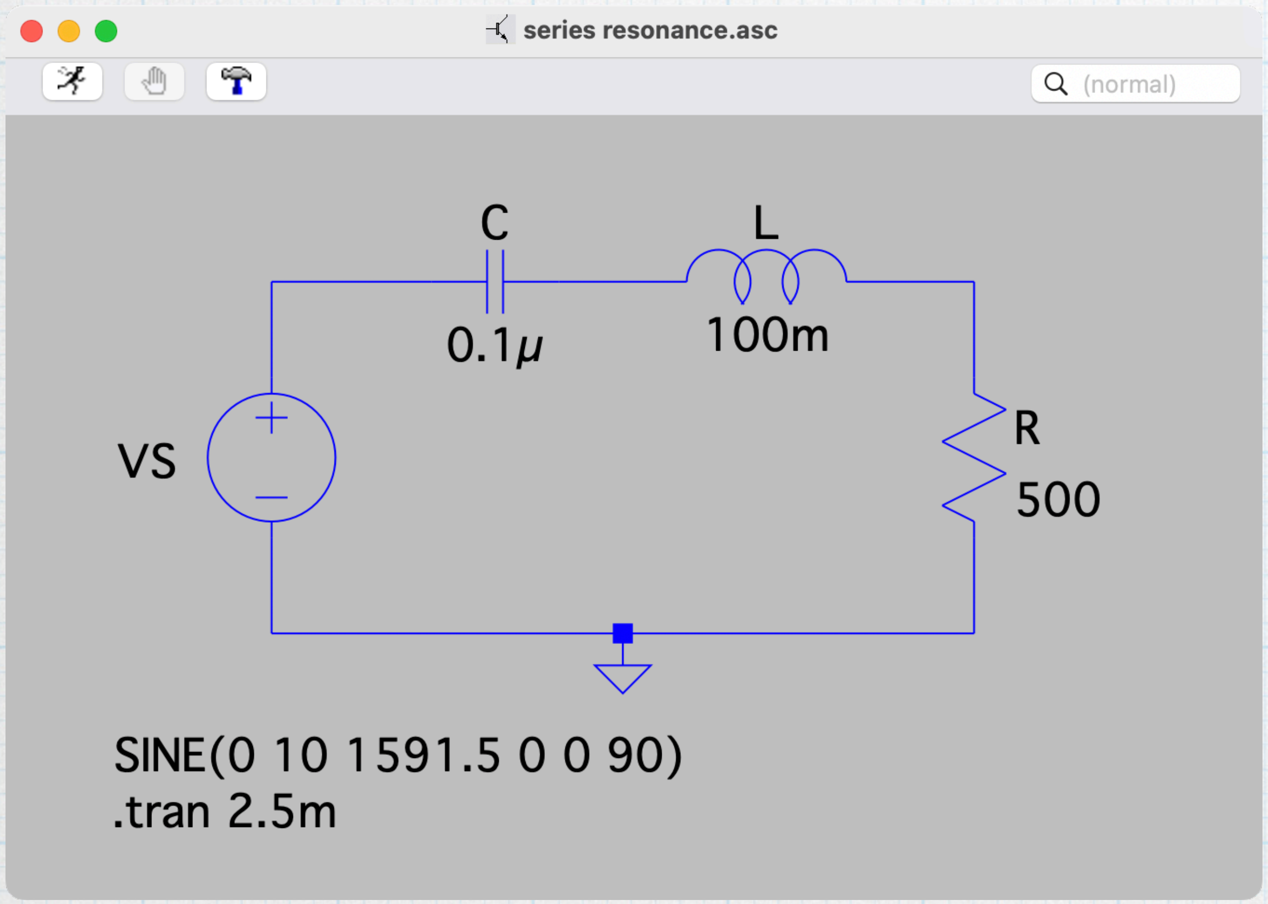
Now we can make a plot of the current as a function of time and see the build-up during the transient.

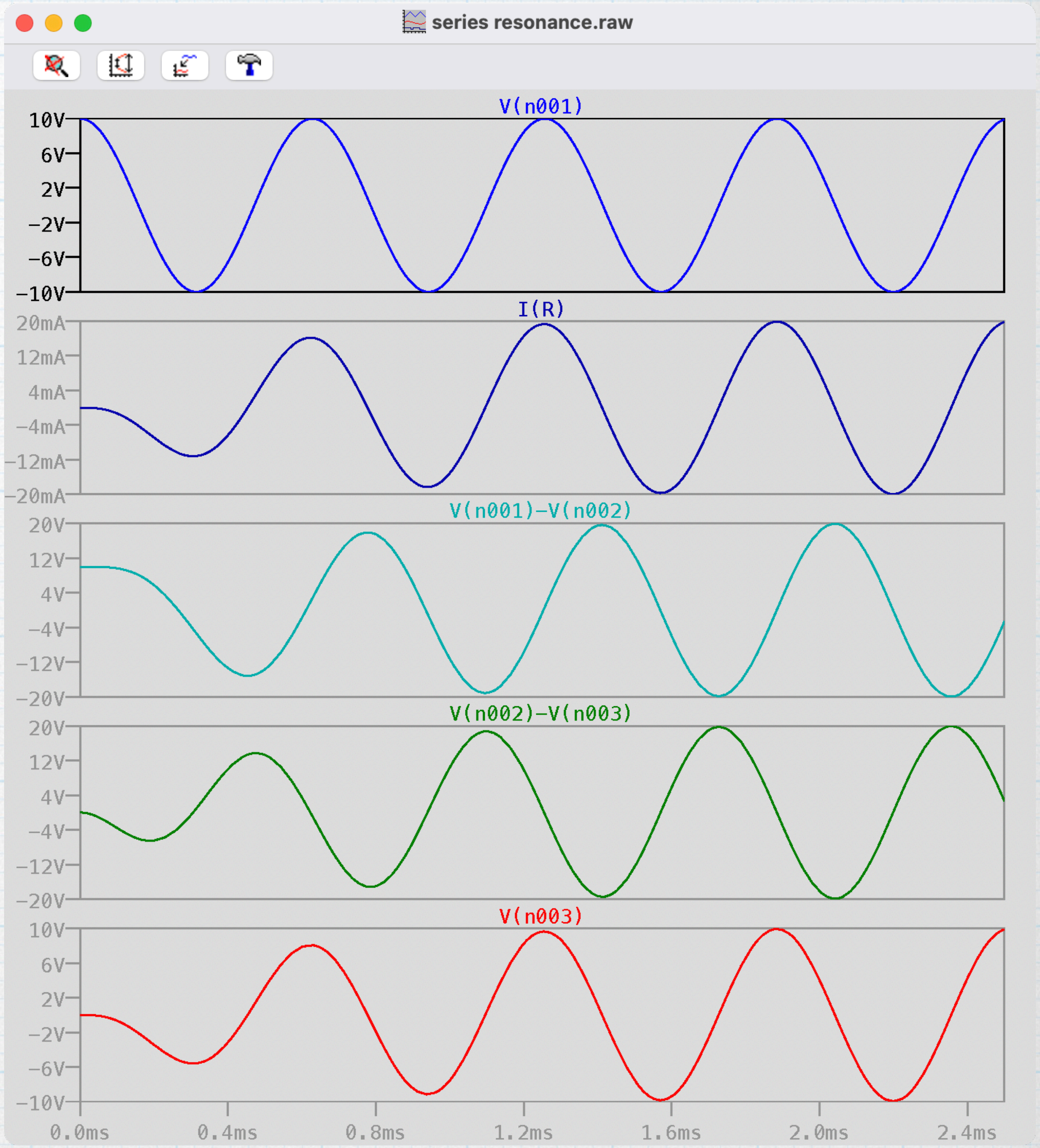
Current build up in the series resonant circuit.



At the start the current is small, but it builds up with each cycle. With a time constant of 0.4 ms, the transient is essentially complete by 2 ms. After that, the current oscillates at the source frequency (which is the resonant frequency of the circuit), with an amplitude of 20 mA, corresponding to the resistor voltage of 10 V and inductor and capacitor voltages of 20 V.

Although it is "fun" to solve the complete equation by hand in order to see the transient build up at resonance, using SPICE allows us to more quickly obtain all the waveforms for different situations.





Note regarding SPICE: The frequency 1591.5 Hz corresponds to 10 krad/s. Also, we added a 90° phase shift to make a cosine rather than a sine in order to match with the hand analysis.

From top to bottom: $v_S(t)$, $i(t)$, $v_C(t)$, $v_L(t)$, and $v_R(t)$.

The slow build-up of the energy in the reactive components is more evident if the time constant of the transient is longer. By reducing R to 50 Ω, the time constant stretches to 4 ms.

Note that reducing the resistance also changes the steady-state current. With R = 50 Ω, the peak current will be 200 mA. The resistance voltage doesn't change, but the peak voltages for the capacitor and inductor are now 200 V!

A 10-V source is producing 200 V on the capacitor and inductor. Remarkable!

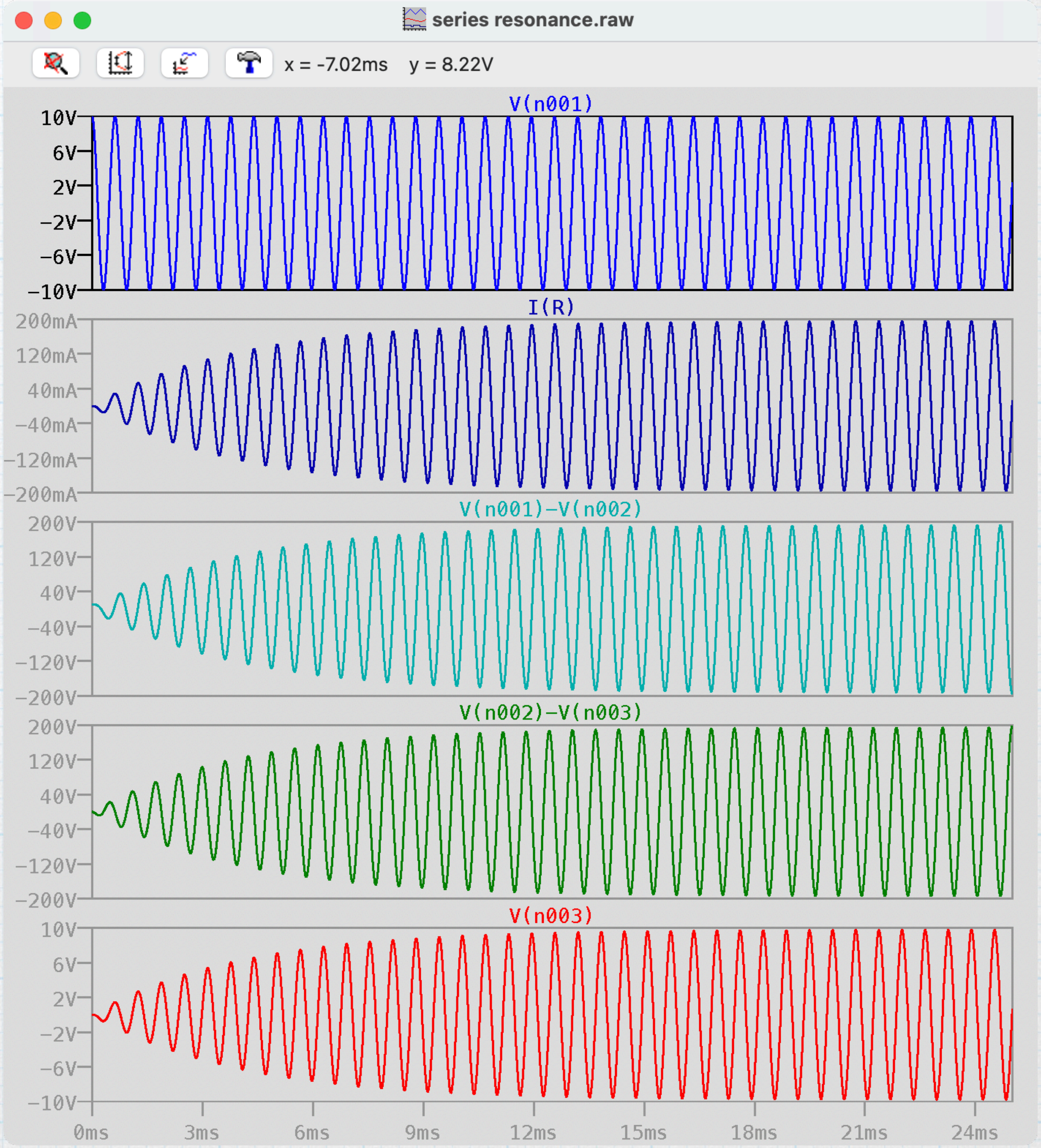
$v_S(t)$

$i(t)$

$v_C(t)$

$v_L(t)$

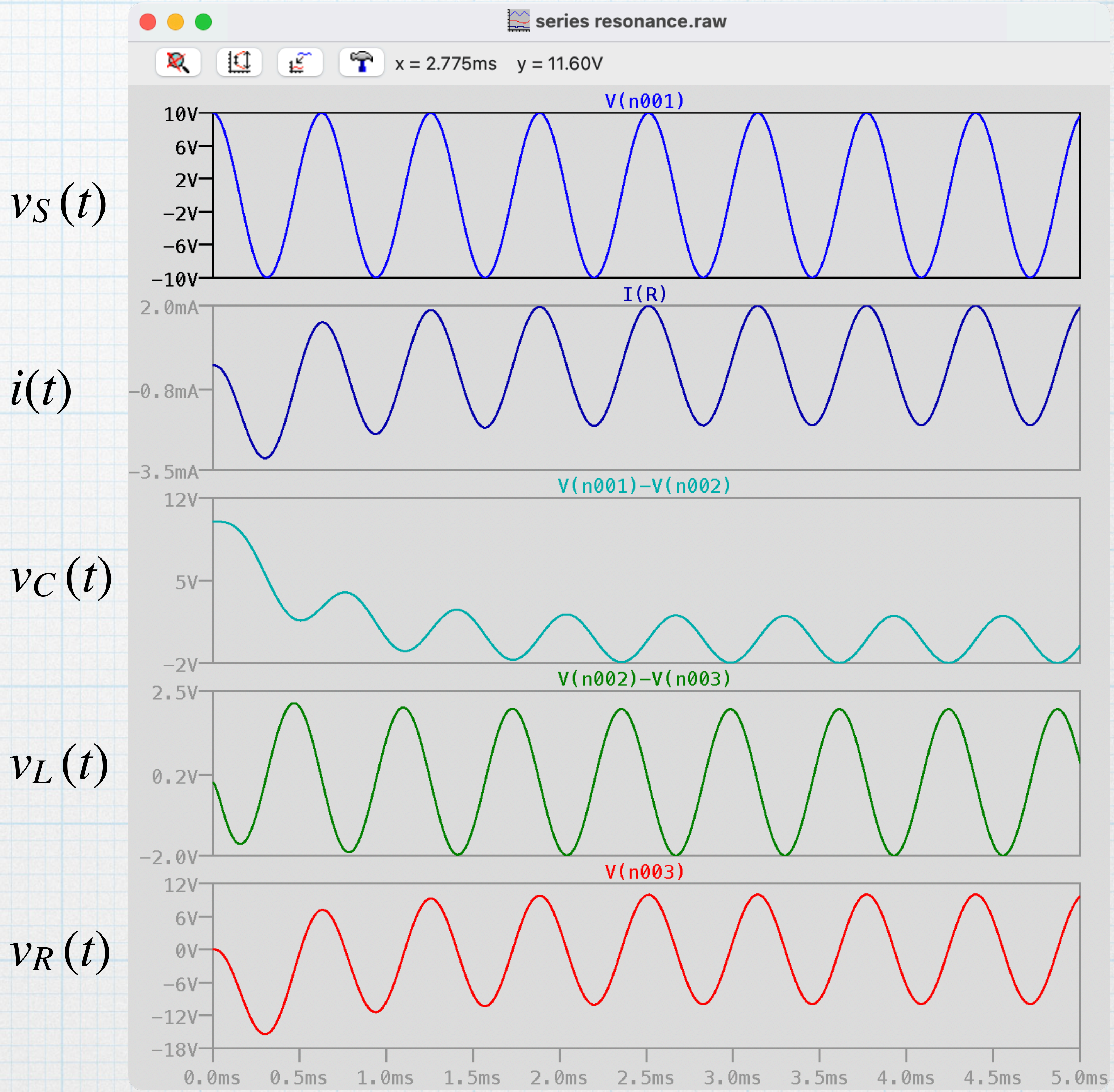
$v_R(t)$



If the resistance is increased, say to $R = 5\ \mathrm{k\Omega}$, so that the transient is no longer underdamped, we still see resonance, $v_R = v_S$ and $v_C = -v_L$.

However, there is no enhancement in the capacitor and inductor voltages. With $R = 5\ \mathrm{k\Omega}$, the peak current at steady-state is 2 mA. The resistor voltage is still 10 V, but the capacitor and inductor peak voltages are now only 2 V.

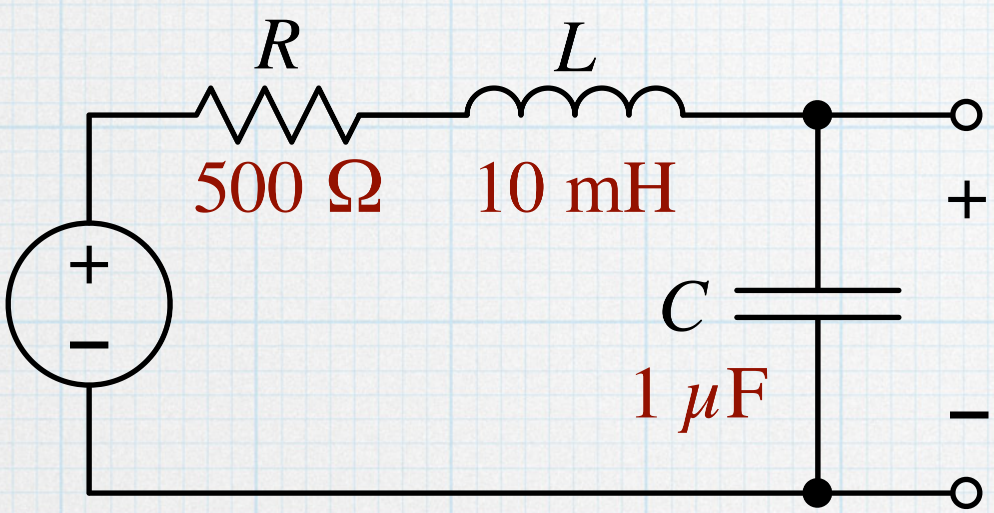
To get enhanced voltages, the R, L, and C must produce an underdamped transient, and the under-damped frequency must be close to the AC source frequency, which is at resonant frequency of the LC combo.

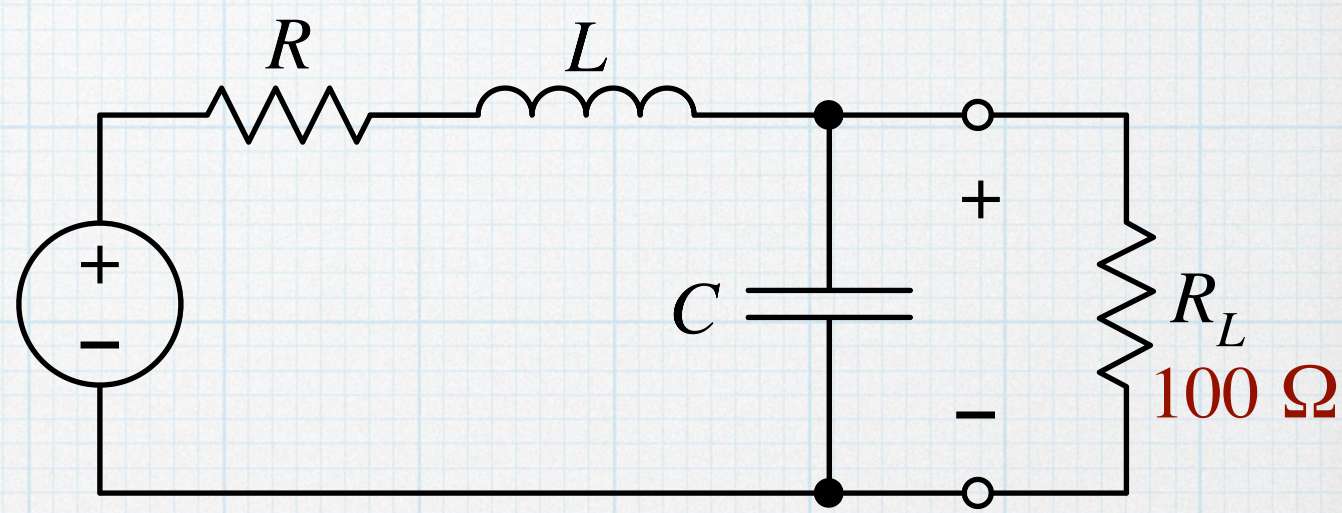


A brief summary of series resonance:

- The condition for series resonance is $\omega_o^2 = LC$. (ω_o is the resonant frequency).
- At resonance, the impedances for the inductor and capacitor cancel. The resistor voltage will match the source in magnitude and phase. The capacitor and inductor voltages will have exactly the same magnitude and be exactly 180° out of phase.
- The series current is determines by the resistance: $\tilde{I}_S = V_m/R = I_m$.
- The the voltage across the reactive components can be found from the current: $\tilde{V}_C = Z_C I_m$ and $\tilde{V}_L = Z_L I_m$.
- If the values of R, L, C are such that $\sqrt{L/C} > R$, then the magnitude of the inductor and capacitor voltages will be bigger than the magnitude of the source. (Exercise: Prove this.) This must necessarily correspond to the transient being fairly underdamped, so that the source frequency and the damped oscillation frequency of the transient "interact" to "pump up" the voltages.
- For bigger values of R, such that the transient is over-damped or only slightly under-damped, there is no enhancement of the inductor and capacitor voltages, but they will still be resonant.

Can we somehow make use of the enhanced voltages seen at resonance? It seems like if we could get that energy out, we might solve the world's energy problems. Unfortunately, this is just not possible. To see the futility of it, consider putting a load resistor across the capacitor in an effort to extract some of that "extra" energy.





$\tilde{V}_S = 10\,\text{V}$ $\qquad$ $\tilde{V}_C = (20\,\text{V})\,e^{-j90^\circ}$ $\qquad$ $\tilde{V}_p = ?$

We immediately see the problem as we start the AC analysis. The extra resistor changes the impedance:

$$\tilde{V}_p = \frac{Z_p}{Z_R + Z_L + Z_p}\tilde{V}_S$$

where $Z_p = Z_C \| Z_{RL} = \dfrac{R_L}{1 + j\omega R_L C}$ is the parallel combination of R_L and C.

Grinding through the details, we find:

$$\tilde{V}_p = \frac{V_m}{\left(1 - \omega^2 LC + \frac{R}{R_L}\right) + j\left(\frac{\omega L}{R_L} + \omega RC\right)}$$

(Work this out as an exercise.) Plugging in numbers:

$$\tilde{V}_p = \frac{10\,\text{V}}{\left(1 - 1 + 5\right) + j\left(10 + 0.5\right)} = \frac{10\,\text{V}}{5 + j10.5}$$

Converting to magnitude and phase: $\tilde{V}_p = \left(0.86\,\text{V}\right) e^{-j64.5^\circ}$.

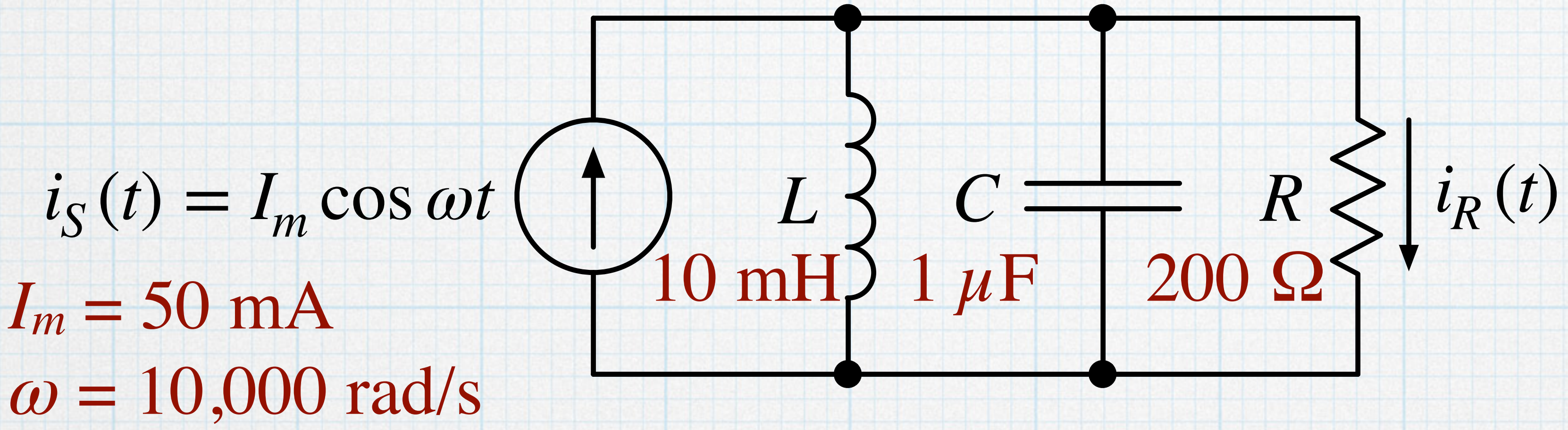
Huh.

Of course, what has happened is that the resistor we added in an attempt to extract the resonance energy has destroyed the resonance. Resonance is a special condition, and the energies being traded back and forth between the inductor and capacitor are necessary to create the special condition. Trying to extract some of the energy messes everything up.

As we know, there are no free lunches.

Parallel resonance

It should come as no surprise that an inductor and a capacitor in parallel will also have exceptional behavior due to resonance. Consider the parallel circuit below — calculate the complex current through the resistor.



Converting everything to complex values and impedances, we can use a current divider to find $\tilde{I}_R$:

$$\tilde{I}_R = \frac{\frac{1}{R}}{\frac{1}{j\omega L} + j\omega C + \frac{1}{R}} I_m = \frac{I_m}{1 + j\left(\omega RC - \frac{R}{\omega L}\right)}$$

Inserting values:

$$\tilde{I}_R = \frac{I_m}{1 + j(2-2)} = I_m$$

The resistor current has exactly same magnitude and phase as the source. Once again, it appears that the inductor and capacitor have effectively disappeared from the circuit.

To see this more clearly, consider the impedance of a parallel LC combination:

$$Z_L \| Z_C = \frac{j\omega L + \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} = \frac{j\omega L}{1 - \omega^2 LC}$$

Obviously, this becomes very interesting if $\omega^2 LC = 1$. In that case the impedance of the parallel combination goes to infinity — it effectively becomes an open circuit and all of source current flows through the resistor.

For any combination of L and C, there will be a frequency, $\omega_o = \frac{1}{\sqrt{LC}}$ where the impedance of the parallel combination goes to infinity. This is the resonance frequency.

It is interesting to note that the formula for parallel resonance is identical to that for series resonance.

At resonance the complex resistor voltage is

$$\tilde{V} = Z_R \tilde{I}_S = R I_m = (200\,\Omega)\left(50\,\text{mA}\right) = 10\,\text{V}.$$

Since everything is in parallel, we can calculate the capacitor and inductor currents:

$$\tilde{I}_L = \frac{\tilde{V}}{Z_L} = \frac{10\,\text{V}}{j100\,\Omega} = -j100\,\text{mA}$$

$$\tilde{I}_C = \frac{\tilde{V}}{Z_C} = \frac{10\,\text{V}}{-j100\,\Omega} = j100\,\text{mA}$$

Again, we see that the magnitudes of the inductor and capacitor currents are greater than the magnitude of the source. It is the exact same story as for the voltages in the series resonant case. If the parallel *RLC* circuit has values that lead to a sufficiently under-damped transient response and the *LC* resonance is close to the damped oscillation frequency of the transient, then the source can "pump up" the current in the *L* and *C* to magnitudes bigger than the source. As long as the inductor and capacitor stay in resonance, this is perfectly fine.